$\boxed{5}$ $(\Omega, \mathcal{F}_T, P)$

$\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\dots,T}$ filtration.

Q a prob. measure $Q > 0$.

Let $L = \{ L(t) := E[\frac{Q}{P} \mid \mathcal{F}_t] \}_{t=0,\dots,T}$

a) Show that $L > 0$ and $L(0) = 1$

b) " " $E_Q[W \mid \mathcal{F}_t] = \dfrac{E[W L(T) \mid \mathcal{F}_t]}{L(t)}$

c) X is $\mathbb{F}$-adapted. X is a $\mathbb{F}$-martingale under Q

$\iff$ XL is a $\mathbb{F}$-martingale under P.

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a) First we prove that if $X > 0$ and $\mathcal{G}$ is an algebra on Ω $\implies$ $E[X \mid \mathcal{G}] > 0$.

Let $A = \{\omega : E[X \mid \mathcal{G}] = 0\}$.

We want to show that $A = \emptyset$

Suppose that $A \neq \emptyset$. Note that $A \in \mathcal{G}$. $E[X \mid \mathcal{G}]$ is $\mathcal{G}$-measurable by construction and then

$$A = E[X \mid \mathcal{G}]^{-1}(0) \in \sigma(E[X \mid \mathcal{G}]) \subseteq \mathcal{G}.$$

Moreover, $E[X \mid \mathcal{G}] \mathbb{1}_A \equiv 0$ and, therefore,

$$0 = E[E[X \mid \mathcal{G}] \mathbb{1}_A] \underset{\substack{\uparrow \\ \text{def. of } E[\cdot \mid \mathcal{G}] \\ \text{and } A \in \mathcal{G}}}{=} E[X \mathbb{1}_A] = \sum \underset{\substack{\vee \\ 0}}{X(\omega)} \underset{\substack{\vee \\ 0}}{P(\omega)} > 0 \,!!$$

This means that $A = \emptyset \Rightarrow E[X \mid \mathcal{G}] > 0$.

On the other hand

$$L(0) = E\left[\frac{Q}{P} \mid \mathcal{F}_0\right] = E\left[\frac{Q}{P}\right] = E_Q[1] = 1$$

b) The candidate for $E_Q[W \mid \mathcal{F}_t]$ is $\dfrac{E[W L(T) \mid \mathcal{F}_t]}{L(t)} =: (*)$

1) $(*)$ is $\mathcal{F}_t$-measurable and is well defined.
$(*)$ the quotient of two $\mathcal{F}_t$-measurable rand. variables $\Rightarrow \mathcal{F}_t$-meas.
and, in addition, is well defined because $L(t) > 0$.

2) Let $A \in \mathcal{F}_t$ since

$$E_Q\left[\frac{E[W L(T) \mid \mathcal{F}_t]}{L(t)} \mathbb{1}_A \right] \stackrel{?}{=} E_Q[W \mathbb{1}_A]$$

$\|$ c.m. of $E_Q[\cdot] = E[\cdot \frac{Q}{P}]$. Since $\mathcal{F} = \mathcal{F}_T \Rightarrow \frac{Q}{P}$ is $\mathcal{F}_T$-meas.
$\Rightarrow \frac{Q}{P} = E[\frac{Q}{P} \mid \mathcal{F}_T] = L(T)$

$$E\left[L(T) \frac{E[W L(T) \mid \mathcal{F}_t]}{L(t)} \mathbb{1}_A \right]$$

$\|$

$$= E\left[E\left[L(T) \overbrace{\frac{E[W L(T) \mid \mathcal{F}_t]}{L(t)} 1\!\!1_A}^{\mathcal{F}_t} \,\Big|\, \mathcal{F}_t \right] \right]$$

law of total expect.

$$= E\left[\cancel{E[L(T) \mid \mathcal{F}_t]} \frac{E[W L(T) \mid \mathcal{F}_t]}{\cancel{L(t)}} 1\!\!1_A \right]$$

MG of $L(t)$

$$= E\left[E[W L(T) 1\!\!1_A \mid \mathcal{F}_t] \right] = E[W L(T) 1\!\!1_A]$$

A is $\mathcal{F}_t$-meas. law of tot. expect.

$$= E_Q[W 1\!\!1_A] \quad \checkmark$$

c) X - $\mathbb{F}$-adapted.

X is $\mathbb{F}$-mart. under Q $\Longleftrightarrow$ XL is $\mathbb{F}$-mart. under P

$\Rightarrow$) XL is $\mathbb{F}$-adapted because it is the product of $\mathbb{F}$-adapted processes.

$$E[X(t+l)L(t+l) \mid \mathcal{F}_t] \overset{?}{=} X(t)L(t)$$

$$E[X(t+l)L(t+l) \mid \mathcal{F}_t] = E[X(t+l)E[L(T) \mid \mathcal{F}_{t+l}] \mid \mathcal{F}_t]$$

$$= E[E[X(t+l)L(T) \mid \mathcal{F}_{t+l}] \mid \mathcal{F}_t]$$

↓ Tower law

$$= E[X(t+l)L(T) \mid \mathcal{F}_t]$$

$$= \frac{E[X(t+l)L(T) \mid \mathcal{F}_t]\, L(t)}{L(t)}$$

$$\overset{b)}{=} E_Q[X(t+l) \mid \mathcal{F}_t]\, L(t)$$

$$\overset{X \text{ is a } Q\text{-martingale}}{=} X(t)L(t)$$

$$(\Leftarrow) \quad E_G[\, X(t+1) \mid \mathcal{F}_t] \overset{?}{=} X(t)$$

$$E_G[\, X(t+1) \mid \mathcal{F}_t] \overset{(i)}{=} \frac{E[\, X(t+1)\, L(T) \mid \mathcal{F}_t]}{L(t)}$$

$X(t+1)$ is $\mathcal{F}_{t+1}$-meas + tower law
$$= \frac{E[\, X(t+1)\, E[L(T) \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t]}{L(t)}$$

def of L
$$= \frac{E[\, X(t+1)\, L(t+1) \mid \mathcal{F}_t]}{L(t)}$$

XL is a mart under P
$$= \frac{X(t) \cdot L(t)}{L(t)} = X(t) \quad \checkmark$$