11 Find a formula for the present value of an infinite stream of payments $C, C(1+g), C(1+g)^2, \ldots$ And for the present value of n of such payments.

Assume there are yearly payments and that they are compounded at a constant rate r.

The present value for n of such payments is given by

$$V(0) = C \sum_{k=1}^{n} \frac{(1+g)^{k-1}}{(1+r)^k} = \frac{C}{1+g} \sum_{k=1}^{n} \frac{1}{\left(\frac{1+r}{1+g}\right)^k}$$

$$= \frac{C}{1+g} PA(n, r_g)$$

where $r_g = \frac{1+r}{1+g} - 1 = \frac{r-g}{1+g}$

Then,

$$V(0) = \frac{C}{1+g} \frac{1-(1+r_g)^{-n}}{r_g}$$

$$= \frac{C}{\cancel{1+g}} \frac{1-\left(\frac{1+r}{1+g}\right)^{-n}}{\frac{r-g}{\cancel{1+g}}} = C \frac{1-\left(\frac{1+g}{1+r}\right)^{n}}{r-g}$$

The present value of an infinite stream of such payments follows by taking limit when $n \to \infty$, which is equal to $C \frac{1}{r-g}$ provided

$$\frac{1+g}{1+r} < 1 \iff \boxed{g < r}$$

12

$\$ 24 \xrightarrow{\text{Invested since 1626 til 2018}} V$

Compounded a) continuously with rate 5%
b) annually

Find V ?

We have $V(0) = 24 \$$, $t = 2018 - 1626 = 392$ and $r = 0'05$

a) Future value with continuous compounding is given by

$$V(t) = V(0)\, e^{rt}$$

which yields

$$V = V(392) = 24 \times e^{0'05 \times 392} = 7'80518 \times 10^9$$

b) Future value with annual compounding is given by

$$V(t) = V(0)\,(1 + r)^t$$

which yields

$$V = V(392) = 24 \times (1 + 0'05)^{392} = 4'85754 \times 10^9$$

13

$$\text{NOK } 9{,}500 \xrightarrow[\text{Continuously compounded at rate } r]{\text{Invested for half a year}} \text{NOK } 10{,}000$$

Find r.

The future value of an investment attracting interest compounded continuously at rate r is given by

$$V(t) = V(0)\, e^{rt},$$

which yields

$$r = \frac{1}{t} \log\left(\frac{V(t)}{V(0)}\right).$$

In this case $t = 0'5$, $V(0'5) = 10{,}000$ and $V(0) = 9{,}500$, and we get

$$r = \frac{1}{0'5} \log\left(\frac{10{,}000}{9{,}500}\right) \approx 0'102587 \simeq 10'26\,\%$$

24 Logarithmic return over 2 months of an investment subject to cont. comp. is 3%. Find the interest rate.

The log return is defined by

$$r(s,t) = \log\left(\frac{V(t)}{V(s)}\right) \qquad 0 \le s < t$$

and the future value with cont. compounding is given by

$$V(t) = V(0)\, e^{rt} \qquad t \ge 0.$$

Hence,

$$r(s,t) = \log\left(\frac{V(0)\, e^{rt}}{V(0)\, e^{rs}}\right) = r(t-s) \qquad 0 \le s < t.$$

which yields

$$r = \frac{r(s,t)}{t-s}$$

In this case, $s=0$, $t = \frac{2}{12} = 1/6$ and $r(0,1/6) = 0'03$

and we obtain

$$r = \frac{0'03}{1/6} = 0'18 = 18\%$$

15 Find the rate for continuous compounding equivalent to monthly compounding at 12%

Equating the corresponding growing factors we get

$$e^{r t} = \left(1 + \frac{r_m}{m}\right)^{t \times m}$$

Continuous comp. at rate r

Periodic compounding with frequency m and rate r_m.

Taking logarithms

$$r \cancel{t} = \cancel{t} m \times \log\left(1 + \frac{r_m}{m}\right)$$

In this case $r_m = 0'12$ and $m = 12$ and we get

$$r = 12 \times \log\left(1 + \frac{0'12}{12}\right) \simeq 0'1194 = 11'94\%$$

16 Present value of an annuity paying C monthly for n years? Express the answer in terms of the effective rate.

Monthly compounding applies, then

$$\left(1+\frac{r}{12}\right)^{12} = 1 + r_e \quad \Rightarrow \quad \frac{r}{12} = (1+r_e)^{\frac{1}{12}} - 1$$

The present value of the annuity is

$$V(0) = C \sum_{k=1}^{12n} \left(1+\frac{r}{12}\right)^{-k}$$

$$= C \, PA\left(12n, \frac{r}{12}\right)$$

$$= C \, \frac{1-\left(1+\frac{r}{12}\right)^{-12n}}{r/12}$$

$$= C \, \frac{1-(1+r_e)^{-n}}{(1+r_e)^{\frac{1}{12}}-1}$$

17 Find the interest rates for annual, semi-annual and continuous compounding implied by a unit zero-coupon bond with $B(0'5,1) = 0'9455$

We have the following formulas for the price at time t of a unit zero-coupon bond.

$$B(t,T) = \left(1 + \frac{r_m}{m}\right)^{-m(T-t)}$$

and

$$B(t,T) = e^{-r_c(T-t)}$$

These yield,

$$r_m = m\left(e^{-\frac{\log(B(t,T))}{m(T-t)}} - 1\right) = m\left(B(t,T)^{-\frac{1}{m(T-t)}} - 1\right)$$

$$r_c = -\frac{\log(B(t,T))}{(T-t)}$$

In this problem we have $t = 0'5$, $T = 1$, $B(0'5,1) = 0'9455$

and we get

- annual $(m=1)$ $\quad r_1 = 1 \times\left((0'9455)^{-\frac{1}{1\times 0'5}} - 1\right) \simeq 0'118605 \simeq 11'86\%$
- semi-annual $(m=2)$ $\quad r_2 = 2\left((0'9455)^{-\frac{1}{2\times 0'5}} - 1\right) \simeq 0'115283 \simeq 11'53\%$
- continuous $\quad r_c = -\frac{\log(0'9455)}{0'5} \simeq 0'112083 \simeq 11'20\%$

18) Find the return on a 75-day investment in zero-coupon bonds if $B(0,1) = 0'89$.

The implied annual compounding rate for a zero-coupon bond is given by

$$r = \frac{F}{B(0,1)} - 1 = \frac{1}{0'89} - 1 = 0'123596 \simeq 12'36\%$$

which is also the effective rate.

The price of the bond after 75 days will be

$$B\left(\frac{75}{365}, 1\right) = B(0,1)\,(1+r)^{\frac{75}{365}}$$

$$= 0'89\,(1+0'1236)^{\frac{75}{365}} \simeq 0'9115$$

The return will be given by

$$R\left(0, \frac{75}{365}\right) = \frac{B\left(\frac{75}{365}, 1\right) - B(0,1)}{B(0,1)} = \frac{0'9115 - 0'89}{0'89}$$

$$\simeq 0'0242 \simeq 2'42\%.$$

$\boxed{19}$ The return on a zero-coupon bond over 6 months is 7%. Find the implied continuous compounding rate

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The return on a zero-coupon bond over the period $[s,t] \subset [0,T]$ is given by

$$R(s,t) = \frac{B(t,T) - B(s,T)}{B(s,T)}$$

The price of the zero-coupon bond at time $t < T$ is given by

$$B(t,T) = e^{-r(T-t)}$$

where r is the implied cont. comp. rate.

Here,

$$R(s,t) = \frac{e^{-r(T-t)} - e^{-r(T-s)}}{e^{-r(T-s)}} = e^{r(t-s)} - 1$$

and $r = \frac{1}{t-s} \log(1 + R(s,t))$

In this exercise, $R(s,t) = 0'07$ and $t - s = \frac{1}{2}$

and

$$r = \frac{1}{1/2} \log(1 + 0'07) \simeq 0'1353 = 13'53\%$$

20 After how many days a bond purchased for $B(0,1) = 0'92$ produce a 5% return

The return of a zero-coupon bond over a period of $[s,t] \subset [0,T]$ is given by

$$R(s,t) = \frac{B(t,T) - B(s,T)}{B(s,T)}$$

and the price at time t is given by

$$B(t,T) = e^{-r(T-t)} \qquad r \text{ imp. cont. comp. rate}$$

Here $s=0$, $T=1$, $B(0,1) = 0'92$ and $R(0,t) = 0'05$.

Hence,

$$0'92 = B(0,1) = e^{-r(1-0)} \Rightarrow r = -\log(0'92)$$
$$= 0'08338 \simeq 8'34\%$$

On the other hand

$$R(0,t) = \frac{e^{-r(1-t)} - B(0,1)}{B(0,1)}$$

which yields

$$t = 1 + \frac{\log(B(0,1)(1+R(0,t)))}{r}$$

Therefore,

$$t = 1 + \frac{\log(0'92(1+0'05))}{0'08338} \simeq 0'5851$$

$$\simeq 213'6 \text{ days}$$