1

- $S(0) = 170$ NOKs stock price.
- $F(0,1) = 180$ NOKs forward price
- $r = 0'08$ interest rate
- Short-selling requires a 30% security deposit attracting interest at $d = 0'04$

Is there an arbitrage opportunity?

Find the highest d for which there is no arbitrage opportunity.

The theoretical forward price is given by

$$F(0,T) = S(0)e^{rT} = 170 \times e^{0'08 \times 1} = 184'159 > 180$$

This suggest that we may build an arbitrage opportunity by entering into a long forward contract and selling short a stock, investing the balance risk-free at rate r. However, we must take into account the cost of short-selling. The candidate strategy is as follows:

- Enter a long forw. contract. (with forw. price 180 NOK) at no cost.
- Sell 1 stock and get $S(0) = 170$ NOK.
- Invest 70% of $S(0)$ risk free at $0'08$ interest rate
- Deposit 30% of $S(0)$ attracting interest at $0'04$.

The initial cost of the strategy is 0.

At time $T = 1$, we use the forward contract to buy a share for $F(0,1)$ and we close the short position in the stock and the position in the money market obtaining

$$-F(0,1) + 0'7\, S(0)e^{r} + 0'3\, S(0)\, e^{d} =$$

$$= -180 + 0'7 \times 170\, e^{0'08} + 0'3 \times 170\, e^{0'04}$$

$$= 1'99251 > 0 \implies \exists$$ arbitrage opportunity.

In order to avoid the arbitrage opportunity the following inequality must hold.

$$0'7 \times S(0)\, e^{r} + 0'3 \times S(0)\, e^{d} \leq F(0,1)$$

, that is,

$$d \leq \log\left(\frac{F(0,1) - 0'7 \times S(0) e^{r}}{0'3 \times S(0)}\right) = \log\left(\frac{180 - 0'7 \times 170\, e^{0'08}}{0'3 \times 170}\right)$$

$$\simeq 0'001740 = 0'174\,\%$$

2

- 1 January 2018 → $t = 0$, Stock price $S(0)$
- 1 April 2018 → $t = 1/4$, $S(1/4) = 0'9\, S(0)$
- 1 October 2018 → $t = 3/4$, $r = 0'06$.

Percentage drop of $F(1/4, 3/4)$ as compared to $F(0, 3/4)$?

The general formula for the forward price is

$$F(t, T) = S(t)\, e^{r(T-t)}$$

We have

$$F(0, 3/4) = S(0)\, e^{0'06 \times 3/4}$$

$$F(1/4, 3/4) = S(1/4)\, e^{0'06 \times 1/2} = 0'9\, S(0)\, e^{0'06 \times 1/2}$$

Therefore, the percentage drop of $F(1/4, 3/4)$ is given by

$$\frac{F(0, 3/4) - F(1/4, 3/4)}{F(0, 3/4)} = \frac{\cancel{S(0)}\, e^{0'06 \times 3/4} - 0'9\, \cancel{S(0)}\, e^{0'06 \times 1/2}}{\cancel{S(0)}\, e^{0'06 \times 3/4}}$$

$$= 1 - 0'9\, e^{0'06\,(1/2 - 3/4)} \simeq 0'113399$$

$$\simeq 11'34\ \%$$

3 r risk free rate

D dividend paid at time t_D.

a) Prove that

$$F(t,T) = \left(S(t) - De^{-r(t_D - t)}\right) e^{r(T-t)} \qquad 0 \leq t \leq t_D$$

b) Dividend paid at a continuous rate r_D. Prove that

$$F(t,T) = S(t)\, e^{(r - r_D)(T-t)}$$

Note that at time t_D, the price of the stock will drop by the amount of the dividend paid.
Hence, this must be accounted by subtracting the present value of the dividend to the current price of the stock $S(t)$. Dividends represent an income.
If the asset has some cost one must add (instead of subtract) the present value of that cost.

a) Suppose that $F(t,T) > \left(S(t) - De^{-r(t_D - t)}\right) e^{r(T-t)}$

The arbitrage strategy is as follows:

<u>At time t</u>

Cost 0
- Take a short forward position with forward price $F(t,T)$.
- Borrow $S(t)$ and buy one share

<u>At time t_D</u>

- Cash the dividend an invest it risk-free

<u>At time T</u>

- Sell the share for $F(t,T)$

- Pay $S(t)e^{r(T-t)}$ to clear the loan.
- Collect $De^{r(T-t_D)}$.

The final balance is

$$F(t,T) - S(t)e^{r(T-t)} + De^{r(T-t_d)} > 0$$

$$\Updownarrow$$

$$F(t,T) > \left(S(t) - De^{r(t_D-t)}\right)e^{r(T-t)} \quad \checkmark$$

If we assume that

$$F(t,T) < \left(S(t) - De^{r(t_D-t)}\right)e^{r(T-t)}$$

one can construct a similar arbitrage strategy.

b) In this case, if the dividend is reinvested in the stock, then an investment in one share at time t will increase to $e^{r_D(T-t)}$ shares at time T. Hence, in order to have one share at time T we should begin with $e^{-r_D(T-t)}$ shares at time t. The following arbitrage strategy uses this fact.

Suppose that $F(t,T) < S(t)e^{(r-r_D)(T-t)}$

Then,

<u>At time t</u>

- Take a long forward position with forw. price $F(t,T)$.
- Sell short a fraction $e^{-r_D(T-t)}$ of a share investing the proceeds $S(t)e^{-r_D(T-t)}$ risk free.

Between time t and T you must pay dividends to the stock owner, raising cash by shorting the stock. Your short position in stock will increase to 1 at time T.

At time T

- Buy one share for $F(t,T)$ and return it to the owner, closing the long forward position and the short position in the stock
- Receive $S(t)\, e^{-r_D(T-t)} \cdot e^{r(T-t)}$ from the risk-free invest.

The final balance will be

$$S(t)\, e^{(r-r_D)(T-t)} - F(t,T) > 0$$

4 $V(t)$ value of a forward contract, with forward price $F(0,T)$ and delivery time T, at time $t \leq T$.

Show that $V(t) > \left(F(t,T) - F(0,T)\right) e^{-r(T-t)}$ (*)

leads to an arbitr. opportunity.

The arbitrage strategy is as follows:

<u>At time t:</u>

- Take a short forward position with forward price $F(0,T)$ by receiving and investing $V(t)$
- Take a long forward position with forward price $F(t,T)$ (at zero cost).

<u>At time T</u>

- Close the forward positions
  - long position $\to$ $S(T) - F(t,T)$
  - short position $\to$ $F(0,T) - S(T)$
- Close the position in the money market $\to$ $V(t)\, e^{r(T-t)}$

The net position is given by

$$V(t)\, e^{r(T-t)} + F(0,T) - F(t,T) > 0$$

Hence, due to (*) we have positive with a zero initial investment, i.e, an arbitrage opportunity.

$\boxed{5}$ $S(0) = 450$ NOK

$r = 0'06$

At $t_d = 1/2$ a dividend of NOK 20 is paid.

For a long forward position with delivery in one year ($T=1$), find its value after 9 months ($t = 3/4$) if

a) $S(3/4) = 490$ NOK

b) $S(3/4) = 450$ NOK.

The forward price of a stock paying a dividend D at $0 < t_d < T$ is

$$F(0,T) = \left(S(0) - e^{-rt_d} D \right) e^{rT}$$

Hence,

$$F(0,1) = \left(450 - 20 \times e^{-0'06 \times 0'5} \right) e^{0'06} \simeq 457'22$$

The formula for the value of a forward contract at time $0 < t < T$ is given by

$$V(t) = \left(F(t,T) - F(0,T) \right) e^{-r(T-t)}$$

where $F(t,T) = S(t)\, e^{r(T-t)}$.

Therefore,

a) $t = 3/4$, $S(3/4) = 490$ and $F(3/4, 1) = 490 \times e^{0'06 \times 1/4} = 497'405$

and $V(3/4) = (497'405 - 457'22)\, e^{0'06 \times 0'25} = 40'79$

b) $t = 3/4$, $S(3/4) = 450$ and $F(3/4, 1) = 450 \times e^{0'06 \times 1/4} \simeq 456'80$

and $V(3/4) = (456'80 - 457'22)\, e^{0'06 \times 0'25} = -0'43$

$\boxed{6}$ r constant. , delivery time $T = 1/4 = 3$ months

$t = 1/365 = 1$ day.

Find $S(1/365)$ such that the marking to market of a futures' contract on this stock with delivery 3 months is zero.

The formula for marking to market is given by

$$f(t,T) - f(0,T) = S(t)e^{r(T-t)} - S(0)e^{rT}$$

If the previous formula is set equal to zero, we get

$$S(t) = \frac{S(0)\,e^{rT}}{e^{r(T-t)}} = S(0)\,e^{rt}$$

That is, the stock must grow at the risk-free rate.

$\boxed{7}$ European put option $K = 360$ NOK, $T = \frac{3}{12} = 1/4$

$P^E = 45$ NOK financed by a loan at $r = 0'12$

comp. cont.

Find $S(T)$ such that the strategy produces a profit of NOK 30

This strategy gives a profit at time T given by

$$P = (K - S(T))^+ - P^E e^{rT}$$

For P to be positive we need $S(T) < K$, in this case $(K - S(T))^+ = K - S(T)$ and we get

$$S(T) = K - P - P^E e^{rT}$$

$$= 360 - 30 - 45 \times e^{0'12 \times 1/4}$$

$$\simeq 283'63 \text{ NOK}.$$

8 European call option $K = 90$ NOK, expiry time $T = 1/2$

$C^E = 8$ NOK financed by a loan at $r = 0'09$ cont. comp.

$$S(T) = \begin{cases} 87 & \text{with prob. } 1/3 \\ 92 & \text{" " } 1/3 \\ 97 & \text{" " } 1/3 \end{cases} \quad \text{NOK}.$$

Find the expected profit of the strategy.

The expect. profit of the strategy is given by

$$E[\text{Profit}] = E\left[(S(T) - K)^+ - C^E e^{rT}\right] = E\left[(S(T)-K)^+\right] - C^E e^{rT}$$

We have that

$$C^E e^{rT} = 8 \times e^{0'09 \times 1/2} \simeq 8'37$$

and

$$E\left[(S(T)-K)^+\right] = E\left[(S(T) - 90)^+\right] = \frac{1}{3}\cancel{(87-90)^+}^{\,0} + \frac{1}{3}(92-90)^+$$

$$+ 1/3\,(97-90)^+$$

$$= \frac{2+7}{3} = \frac{9}{3} = 3$$

Therefore,

$$E[\text{Profit}] = 3 - 8'37 = -5'37$$

9

$S(0) = 156$

$C^E = 28'3$. expiry time $T = 1/4 = 3$ months

strike price $K = 150$

$r = 0'0672$

Find P^E.

The put call parity for European options states

$$C^E - P^E = S(0) - Ke^{-rT}$$

Hence,

$$P^E = C^E - S(0) + Ke^{-rT}$$

$$= 28'3 - 156 + 150\, e^{-0'0672 \times 1/4}$$

$$\simeq 19'8$$

10

$C^E = 50'9$
$P^E = 77'8$ $\Big\} \Rightarrow$ Expiry time $T = 1/2$ (6 months)
Strike price $K = 240$ NOK

$S(0) = 203'7$ NOK and $r = 0'0748$

Find an arbitrage opportunity.

The arbitrage opportunity must follow from a violation of the put-call parity

$$C^E - P^E = S(0) - Ke^{-rT}$$

We have that

$$C^E - P^E = 50'9 - 77'8 = -26'9$$
$$\vee$$
$$S(0) - Ke^{-rT} = 203'7 - 240\,e^{-0'0748 \times 1/2} \simeq -27'49$$

Hence, we have that

$$C^E - P^E - S(0) + Ke^{-rT} > 0$$

The arbitrage strategy is as follows:

- Sell a call for $C^E = 50'9$
- Buy a put for $P^E = 77'8$
- Buy a share for $S(0) = 203'7$
- Borrow $S(0) - C^E + P^E = 230'6$ at rate $0'0748$

At $T = 1/2$

- Sell the share for $K = 240$ by exercising the put if $S(1/2) < 240$ or closing the call if $S(1/2) > 240$.
- Repay the loan which amounts to $230'6 \times e^{0'0748 \times 1/2} = 239'39$

This gives a sure profit of $240 - 239'39 = 0'61$ NOK.

11 Prove that

a) $0 \leq C^E \leq C^A$

b) $0 \leq P^E \leq P^A$

---

a) Suppose $C^E < 0$.

- Buy a call option for C^E, receiving $-C^E > 0$
- Invest risk free $-C^E$

At expiring time T we get the strictly positive amount $-C^E e^{rT}$ from the risk free investment and a nonnegative amount $(C^E - K)^+$. Hence an arbitrage.

The argument for showing $P^E \geq 0$ is the same.

For showing that $C^E \leq C^A$, suppose that $C^E > C^A$.

Then, at time 0.

- Sell the European call
- Buy the American call
- Invest the strictly positive balance $C^E - C^A$.

At time T:

- Close the European call getting $-(S(T)-K)^+$ (If it is exercised)
- " " American call getting $(S(T)-K)^+$
- Close the risk free investment.

The final balance is

$$-\cancel{(S(T)-K)^+} + \cancel{(S(T)-K)^+} + (C^E - C^A)e^{rT} > 0$$

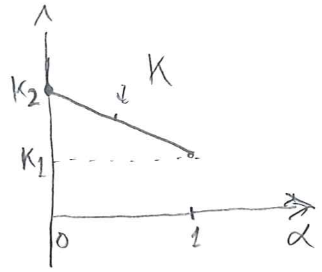
b) It is analogous to a). The strategy is to sell the expensive option and buy the cheapest one, investing the difference risk free.

12 Let $(x)^+ = \max(x, 0)$. Let $0 \leq K_1 < K_2$, $\alpha \in (0,1)$ and $K = \alpha K_1 + (1-\alpha) K_2$. Prove that

$$(x - K)^+ \leq \alpha (x - K_1)^+ + (1-\alpha)(x - K_2)^+ \quad (*)$$

Note that

$$K(\alpha) = -(K_2 - K_1)\,\alpha + K_2$$



Hence $K(\alpha) \in (K_1, K_2)$ if $\alpha \in (0,1)$

- Case $\boxed{x \leq K_1}$: Then $x \leq K < K_2$ and $(*)$ reads $0 \leq 0$ ✓
- Case $\boxed{K_1 < x \leq K}$ Then $x < K_2$ and $(*)$ reads

  $0 \leq \alpha(x - K_1) + 0$, which holds because $x > K_1$.
- Case $\boxed{K < x \leq K_2}$. Then $x > K_1$ and $(*)$ reads

$$x - \underbrace{(\alpha K_1 + (1-\alpha) K_2)}_{K} \leq \alpha (x - K_1)$$

$$\Updownarrow$$

$$(1-\alpha)(x - K_2) \leq 0$$

  which holds because $(1-\alpha) > 0$ and $x \leq K_2$.
- Case $\boxed{x > K_2}$. Then $x > K > K_1$ and

$$\begin{aligned}(x - \underbrace{(\alpha K_1 + (1-\alpha) K_2)}_{K})^+ &= x - \alpha K_1 - (1-\alpha) K_2 \\ &= (\alpha + 1 - \alpha) x - \alpha K_1 - (1-\alpha) K_2 \\ &= \alpha (x - K_1) + (1-\alpha)(x - K_2) \\ &= \alpha (x - K_1)^+ + (1-\alpha)(x - K_2)^+\end{aligned}$$

The function $(x)^+ = \max(0, x)$ is convex because the maximum of two convex functions is convex. The const. function 0 and the linear function x are convex. Affine functions are convex. If f and g are convex functions and g is non-decreasing, then $h(x) = g(f(x))$ is convex. Take $f(K) = x - K$ and $g(y) = (y)^+ \Rightarrow (x-K)^+$ is convex in K which yields $(*)$.

13 Prove that, on a stock paying no dividends, one has

a) $(S(0)-K)^+ \le C^A < S(0)$

b) $(K-S(0))^+ \le P^A < K$

---

a) It trivially follows from the fact that $C^E = C^A$ for a stock paying no dividends and the analogous inequalities for a European call.

b) We already know that $P^A \geq 0$ so we only need to show that $P^A \geq K - S(0)$ and $P^A < K$.

But the value of an American put must always be greater than its intrinsic value which is equal to $K - S(0)$.

To prove $P^A < K$, note that the put-call parity estimates for American options ($C^A - P^A \geq S(0) - K$) implies that

$$P^A \le C^A - S(0) + K < K$$

because $C^A - S(0) < 0$.

14

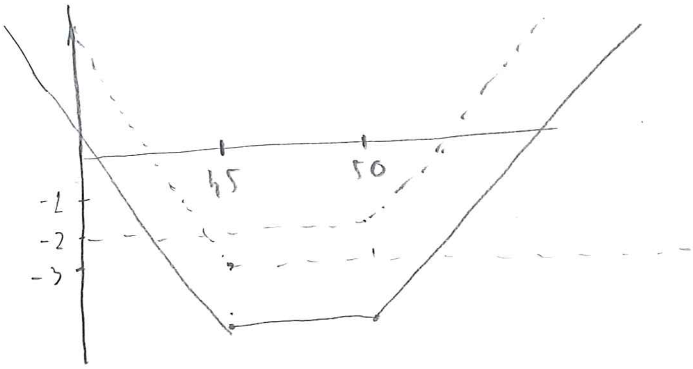
$C^E = 2$ with $K_1 = 50$

$P^E = 3$ with $K_2 = 45$

How a strangle is build and what is the pattern of profits?

A strangle is built by buying the two calls and it has an initial cost of $C^E + P^E = 5$.

The payoff diagram is as follows (I assume $r = 0$).



This can be summarized in the following table.

Stock price S_T	Profit
$S_T < 45$	$(45 - S_T) - 5 = 40 - S_T$
$45 < S_T < 50$	-5
$S_T > 50$	$(S_T - 50) - 5 = S_T - 55$

15

long strangle
short straddle } => Trading position?

Assuming the strike price in the straddle is halfway between the two strike prices of the strangle

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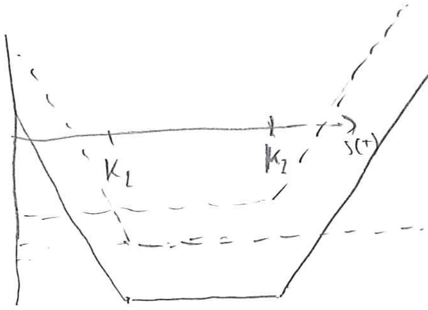
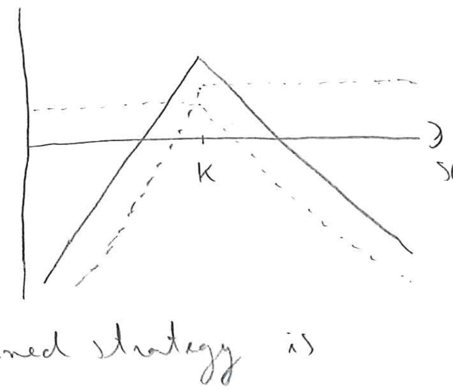
A long strangle with strike prices K_1 and K_2, $K_1 < K_2$ is generated by buying a call option with strike K_2 and buying a put option with strike K_1.



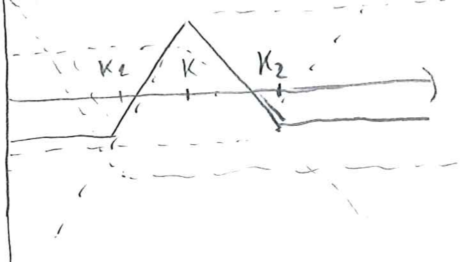
S_T	Profit
$S_T < K_1$	$(K_1 - S_T) - (C^E(K_2) + P^E(K_1))e^{rT}$
$K_1 < S_T < K_2$	$-(C^E(K_2) + P^E(K_1))e^{rT}$
$S_T > K_2$	$(S_T - K_2) - (C^E(K_2) + P^E(K_1))e^{rT}$

A short straddle with strike $K = \frac{K_1 + K_2}{2}$ is generated by selling a call and a put with the same strike K



S_T	Profit
$S_T < K$	$-(K - S_T) + (C^E(K) + P^E(K))e^{rT}$
$S_T > K$	$-(S_T - K) + (C^E(K) + P^E(K))e^{rT}$

The combined strategy is



S_T	Profit
$S_T < K_2$	$-\frac{K_2 - K_1}{2} - (C^E(K_2) - C^E(K) + P^E(K_1) - P^E(K))e^{rT}$
$K_1 < S_T < K$	$-\left(\frac{K_1+K_2}{2} - S_T\right) - (C^E(K_2) - C^E(K) + P^E(K_1) - P^E(K))e^{rT}$
$K < S_T < K_2$	$-\left(S_T - \frac{K_1+K_2}{2}\right) - (C^E(K_2) - C^E(K) + P^E(K_1) - P^E(K))e^{rT}$
$S_T > K_2$	$-\frac{K_2 - K_1}{2} - (C^E(K_2) - C^E(K) + P^E(K_1) - P^E(K))e^{rT}$

That is a butterfly spread. Show that ↗ is negative.