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Single Period Financial Markets

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1. Recall that the discounted price process $S^* = \left\{ \left(S_1^*(t), ..., S_N^*(t) \right)^T \right\}_{t=0,1}$ is defined by $S_n^*(t) = S_n(t)/B(t)$, $n = 1, ..., N$, where $B = \{B(t)\}_{t=0,1}$ is the bank account process. Given a trading strategy $H = (H_1, ..., H_N)^T$ the discounted value process is defined to be

$$V^*(t) = H_0 + \sum_{n=1}^{N} H_n S_n^*(t), \qquad t = 0, 1,$$

and the gains process G^* by the random variable $G^* = \sum_{n=1}^{N} H_n \Delta S_n^*$, where $\Delta S_n^* = S_n^*(1) - S_n^*(0)$. Verify that

   (a) $V^*(t) = V(t)/B(t), t = 0, 1.$

   (b) $V^*(1) = V^*(0) + G^*.$

2. Specify V, V^*, G and G^* in the following models

   (a) $K = 3, N = 1, r = 1/9, S_1(0) = 5, S_1(1) = (20/3, 40/9, 30/9)^T$.

   (b) $K = 3, N = 2, B(0) = 1, S_1(0) = 5, S_2(0) = 10,$

$$S(1, \Omega) = \begin{pmatrix} 10/9 & 60/9 & 40/3 \\ 10/9 & 60/9 & 80/9 \\ 10/9 & 40/9 & 80/9 \end{pmatrix}.$$

   (c) $K = 4, N = 2, B(0) = 1, S_1(0) = 5, S_2(0) = 10,$

$$S(1, \Omega) = \begin{pmatrix} 10/9 & 60/9 & 40/3 \\ 10/9 & 60/9 & 80/9 \\ 10/9 & 40/9 & 80/9 \\ 10/9 & 20/9 & 120/9 \end{pmatrix}.$$

3. Consider the model with $K = 3, N = 2, B(0) = 1, S_1(0) = 4, S_2(0) = 7$,

$$S(1, \Omega) = \begin{pmatrix} 1 & 8 & 10 \\ 1 & 6 & 8 \\ 1 & 3 & 4 \end{pmatrix}.$$

Show that there exist dominant trading strategies and that the law of one price holds.

4. Consider the model with $K = 3, N = 2, B(0) = 1, S_1(0) = 5, S_2(0) = 10$,

$$S(1, \Omega) = \begin{pmatrix} 10/9 & 60/9 & 40/3 \\ 10/9 & 60/9 & 80/9 \\ 10/9 & 40/9 & 80/9 \end{pmatrix}.$$

Show that there are no dominant trading strategies but there exists an arbitrage opportunity.

5. Show that

$$W = \left\{X \in \mathbb{R}^K : X = G^* \quad \text{for some trading strategy } H\right\},$$
$$W^{\perp} = \left\{Y \in R^K : X^T Y = 0, X \in W\right\},$$

are linear subspaces.

6. Specify W and $W^{\perp}$ in the case of
   (a) $K = 2, N = 1, r = 1/9, S_1(0) = 5, S(1) = (20/3, 40/9)^T$.
   (b) Exercise 2. (a).
   (c) Exercise 2. (c).

7. Determine either all the risk neutral probability measures or all the arbitrage opportunities in the case of exercise 2. (c).

8. Suppose $K = 2; N = 1$ and the interest rate is $r \geq 0$. Also, suppose $S_1(0) = 1, S_1(1, \omega_1) = u, S_1(1, \omega_2) = d$, where $u > d > 0$. For what values of r, u and d there exist a risk neutral probability measure? Give this measure. For the complementary values of these parameters, describe all the arbitrage opportunities.

9. In the case of exercise 6. (a), what is the price of a put option with exercise price 5, that is, the price of the contingent claim $X = (5 - S_1(1))^+$? What trading strategy generates this contingent claim?

10. (**Put-Call parity**) Suppose the interest rate is $r \geq 0$, and let C and P denote the prices of a call and put option, respectively, both having the same strike K. Show that either both are attainable or neither is attainable. Use risk neutral valuation to show that in the former case one has

$$C - P = S(0) - K(1 + r)^{-1}.$$

11. Explain why the model in exercise 2. (c) is not complete. Characterize the set of all attainable contingent claims. Compute $V_+(X)$ and $V_-(X)$ for $X = (40, 30, 20, 10)^T$.

12. Use the fact that a contingent claim X is attainable if and only if $\mathbb{E}_Q[X/B(1)]$ is constant when varying $Q \in M$ to check whether there are any values of the exercise price K such that the call option is attainable for the model in exercise 2 (a). Similarly, specify which put options are attainable.

13. Recall that $P^+ = \{X \in R^K : X_1 + \cdots + X_K = 1, X_1 > 0, ..., X_K > 0\}$. Assume that $\dim(W^\perp) = J$. Prove that you can choose $Q_j \in W^\perp \cap P^+, j = 1, ..., J$ to be linearly independent vectors. Compute Q_j for the model in exercise 2. (c).

14. Suppose the model in exercise 6. (a). Using the direct approach, solve the optimal portfolio problem for the initial wealth $v \geq 0$ and the probability $P(\omega_1) = p$ under the utility functions
    (a) $U(u) = \log(u)$,
    (b) $U(u) = -\exp(-u)$,
    (c) $U(u) = \gamma^{-1}u^\gamma$, where $-\infty < \gamma < 1, \gamma \neq 0$.

15. Suppose $U(u) = \log(u)$. Show that the inverse function $I(i) = i^{-1}$, the optimal Lagrange multiplier $\hat{\lambda} = v^{-1}$, the optimal attainable wealth is $\hat{W} = vL^{-1}B(1)$, and the optimal objective value is $\mathbb{E}\left[U\left(\hat{W}\right)\right] = \log(v) - \mathbb{E}\left[\log\left(LB^{-1}(1)\right)\right]$. Compute these expressions and solve for the optimal trading strategy in the model in exercise 14 with $p = 3/5$.

16. Suppose $U(u) = \gamma^{-1}u^\gamma$, where $-\infty < \gamma < 1, \gamma \neq 0$. Show that the inverse function $I(i) = i^{-\frac{1}{1-\gamma}}$, the optimal Lagrange multiplier

$$\hat{\lambda} = v^{-(1-\gamma)}\left(\mathbb{E}\left[\left(LB^{-1}(1)\right)^{-\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma},$$

the optimal attainable wealth

$$\hat{W} = \frac{v\left(LB^{-1}(1)\right)^{-\frac{1}{1-\gamma}}}{\mathbb{E}\left[\left(LB^{-1}(1)\right)^{-\frac{\gamma}{1-\gamma}}\right]},$$

and the optimal objective value $\mathbb{E}\left[U\left(\hat{W}\right)\right] = \lambda v/\gamma$. Compute these expressions and solve for the optimal trading strategy in the model in exercise 14 with $p = 3/5$.