Professor: *S. Ortiz-Latorre*

Multiperiod Financial Markets

---

1. Let $\mathbb{F} = \{\mathcal{F}\}_{t=0,...,T}$ be a filtration on a finite probability space $(\Omega, \mathcal{F}, P)$. Show that a $\mathbb{F}$-adapted process $M = \{M(t)\}_{t=0,...,T}$ is a martingale if and only if

$$\mathbb{E}[M(t+1)|\mathcal{F}_t] = M(t), \quad t = 0, ..., T-1.$$

   Show that if $M = \{M(t)\}_{t=0,...,T}$ is martingale then $\mathbb{E}[M(t)] = \mathbb{E}[M(0)]$,$t = 0, ..., T$.

2. Let $\mathbb{F} = \{\mathcal{F}\}_{t=0,...,T}$ be a filtration on a finite probability space $(\Omega, \mathcal{F}, P)$. Show that if $H = \{H(t)\}_{t=1,...,T}$ is a $\mathbb{F}$-predictable process and $S = \{S(t)\}_{t=0,...,T}$ is a $\mathbb{F}$-adapted martingale, then

$$M(t) = \sum_{u=1}^{t} H(u)(S(u) - S(u-1)), \quad t = 0, ..., T,$$

   is a $\mathbb{F}$-adapted martingale. By convention we set $M(0) = 0$.

3. Let $\mathbb{F} = \{\mathcal{F}\}_{t=0,...,T}$ be a filtration and X a random variable, both defined on a finite probability space $(\Omega, \mathcal{F}, P)$. Show that the process $Y = \{Y(t) = \mathbb{E}[X|\mathcal{F}_t]\}_{t=0,...,T}$ is a $\mathbb{F}$-martingale.

4. Show that if X is a random variable on a finite probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{G}$ is an algebra on Ω and $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function then

$$\varphi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\varphi(X)|\mathcal{G}].$$

   Moreover, show that if $Y = \{Y(t)\}_{t=0,...,T}$ is a $\mathbb{F}$-martingale then $Z = \{Z(t) = \varphi(Y(t))\}_{t=0,...,T}$ is a $\mathbb{F}$-submartingale.

5. Let $\mathbb{F} = \{\mathcal{F}\}_{t=0,\ldots,T}$ be a filtration on a finite probability space $(\Omega, \mathcal{F}_T, P)$. Let Q be another probability measure on Ω that is equivalent to P, i.e., such that $Q > 0$. Define the process $L = \left\{L(t) = \mathbb{E}\left[\frac{Q}{P}|\mathcal{F}_t\right]\right\}_{t=0,\ldots,T}$. (Recall that, given a random variable Y, $\mathbb{E}_Q[Y] = \mathbb{E}\left[Y\frac{Q}{P}\right]$)

   (a) Show that L is strictly positive with $L(0) = 1$.

   (b) Let W be a random variable. Then $\mathbb{E}_Q[W|\mathcal{F}_t] = \frac{\mathbb{E}[WL(T)|\mathcal{F}_t]}{L(t)}$.

   (c) Let $X = \{X(t)\}_{t=0,\ldots,T}$ be a $\mathbb{F}$-adapted process. Show that X is a $\mathbb{F}$-martingale under Q if and only if $Z = \{Z(t) = L(t)X(t)\}_{t=0,\ldots,T}$ is a $\mathbb{F}$-martingale under P.

6. Consider a 2-period market with $\Omega = \{\omega_1, \ldots, \omega_5\}$, $r = 0$, and one risky security with $S(0) = 6$,

$$S(1) = (5,5,5,7,7)^T, \quad S(2) = (3,4,8,6,8)^T.$$

   The filtration is the one generated by the risky security. Show that the set of all the martingale measures is

$$\mathbb{M} = \left\{Q \in \mathbb{R}^5 : Q = \left(\frac{\lambda}{2}, \frac{3-5\lambda}{8}, \frac{1+\lambda}{8}, \frac{1}{4}, \frac{1}{4}\right), \quad 0 < \lambda < 3/5\right\}.$$

   Characterize the set of attainable contingent claims in this market. Find the interval of arbitrage free prices for the contingent claim $X = (2,1,1,2,3)^T$.

7. Consider a 2-period market with $\Omega = \{\omega_1, \ldots, \omega_4\}$, $P = (1/4, 1/4, 1/4, 1/4)^T$, $r = 0$, and one risky security with $S(0) = 5$,

$$S(1) = (8,8,4,4)^T, \quad S(2) = (9,6,6,3)^T.$$

   Compute the optimal attainable wealth, the optimal expected utility, and the optimal trading strategy under the utility function $U(u) = -u^{-1}$.