An Introduction to Mathematical Finance Exercises 3
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Professor: *S. Ortiz-Latorre*

Single Period Financial Markets

1.

(a) Note that for $t=0,1$.

$$V^*(t) = H_0 + \sum_{n=1}^{N} H_n S_n^*(t) = H_0 \frac{B(t)}{B(t)} + \sum_{n=1}^{N} H_n \frac{S_n(t)}{B(t)}$$
$$= \frac{1}{B(t)} \left(H_0 B(t) + \sum_{n=1}^{N} H_n S_n(t) \right) = V(t)/B(t).$$

(b) Note that

$$V^*(1) = H_0 + \sum_{n=1}^{N} H_n S_n^*(1)$$
$$= H_0 + \sum_{n=1}^{N} H_n S_n^*(0) + \sum_{n=1}^{N} H_n S_n^*(1) - \sum_{n=1}^{N} H_n S_n^*(0)$$
$$= V^*(0) + \sum_{n=1}^{N} H_n (S_n^*(1) - S_n^*(0)) = V^*(0) + G^*.$$

2. The solutions are:

(a) $V(0) = V^*(0) = H_0 + 5H_1$,

$$V(1) = \begin{pmatrix} \frac{10}{9}H_0 + \frac{20}{3}H_1 \\ \frac{10}{9}H_0 + \frac{40}{9}H_1 \\ \frac{10}{9}H_0 + \frac{10}{3}H_1 \end{pmatrix}, V^*(1) = \begin{pmatrix} H_0 + 6H_1 \\ H_0 + 4H_1 \\ H_0 + 3H_1 \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{1}{9}H_0 + \frac{5}{3}H_1 \\ \frac{1}{9}H_0 - \frac{5}{9}H_1 \\ \frac{1}{9}H_0 - \frac{5}{3}H_1 \end{pmatrix}, G^* = \begin{pmatrix} H_1 \\ -H_1 \\ -2H_1 \end{pmatrix},$$

(b) $V(0) = V^*(0) = H_0 + 5H_1 + 10H_2$,

$$V(1) = \begin{pmatrix} \frac{10}{9}H_0 + \frac{60}{9}H_1 + \frac{120}{9}H_2 \\ \frac{10}{9}H_0 + \frac{60}{9}H_1 + \frac{80}{9}H_2 \\ \frac{10}{9}H_0 + \frac{40}{9}H_1 + \frac{80}{9}H_2 \end{pmatrix}, V^*(1) = \begin{pmatrix} H_0 + 6H_1 + 12H_2 \\ H_0 + 6H_1 + 8H_2 \\ H_0 + 4H_1 + 8H_2 \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{1}{9}H_0 + \frac{15}{9}H_1 + \frac{30}{9}H_2 \\ \frac{1}{9}H_0 + \frac{15}{9}H_1 - \frac{10}{9}H_2 \\ \frac{1}{9}H_0 - \frac{5}{9}H_1 - \frac{10}{9}H_2 \end{pmatrix}, G^* = \begin{pmatrix} H_1 + 2H_2 \\ H_1 - 2H_2 \\ -H_1 - 2H_2 \end{pmatrix}.$$

(c) $V(0) = V^*(0) = H_0 + 5H_1 + 10H_2$,

$$V(1) = \begin{pmatrix} \frac{10}{9}H_0 + \frac{60}{9}H_1 + \frac{120}{9}H_2 \\ \frac{10}{9}H_0 + \frac{60}{9}H_1 + \frac{80}{9}H_2 \\ \frac{10}{9}H_0 + \frac{40}{9}H_1 + \frac{80}{9}H_2 \\ \frac{10}{9}H_0 + \frac{20}{9}H_1 + \frac{120}{9}H_2 \end{pmatrix}, V^*(1) = \begin{pmatrix} H_0 + 6H_1 + 12H_2 \\ H_0 + 6H_1 + 8H_2 \\ H_0 + 4H_1 + 8H_2 \\ H_0 + 2H_1 + 12H_2 \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{1}{9}H_0 + \frac{15}{9}H_1 + \frac{30}{9}H_2 \\ \frac{1}{9}H_0 + \frac{15}{9}H_1 - \frac{10}{9}H_2 \\ \frac{1}{9}H_0 - \frac{5}{9}H_1 - \frac{10}{9}H_2 \\ \frac{1}{9}H_0 - \frac{25}{9}H_1 - \frac{30}{9}H_2 \end{pmatrix}, G^* = \begin{pmatrix} H_1 + 2H_2 \\ H_1 - 2H_2 \\ -H_1 - 2H_2 \\ -3H_1 + 2H_2 \end{pmatrix}.$$

3. To show that there exist a dominating trading strategy $H = (H_0, H_1, H_2)^T$ the following conditions must be satisfied

$$0 = V(0) = H_0 + 4H_1 + 7H_2,$$

$$0 < V(1) = S(1,\Omega)H = \begin{pmatrix} H_0 + 8H_1 + 10H_2 \\ H_0 + 6H_1 + 8H_2 \\ H_0 + 3H_1 + 4H_2 \end{pmatrix}.$$

One gets that the dominating trading strategies are of the following form

$$H = (-4H_1 - 7H_2, H_1, H_2)^T, \quad H_1 > 0, H_2 \in \left(-\frac{4H_1}{3}, -\frac{H_1}{3}\right).$$

To show that the law of one price holds it suffices to show that for arbitrary $X \in \mathbb{R}^3$ the equation $X = S(1,\Omega)H$ has a unique solution. This solution is

$$H_0 = -X_2 + 2X_3, \quad H_1 = 2X_1 - 3X_2 + X_3, \quad H_2 = -X_3 + \frac{5}{2}X_2 - \frac{3}{2}X_1.$$

4. To show that there are no dominating trading strategies but there are arbitrage opportunities it suffices to solve

$$Q^T S^*(1,\Omega) = (1, S_1^*(0), S_2^*(0)),$$

for $Q \in \mathbb{R}^3$. The solution is $Q = \left(\frac{1}{2}, 0, \frac{1}{2}\right)^T$. Hence, Q is a linear pricing measure, which is equivalent to say that there are no dominating trading strategies. This also shows that there are no risk neutral measures in this market (because they solve the same equation) and we can conclude (by the first fundamental theorem of asset pricing) that the market contains arbitrage opportunities. One can actually show that the only arbitrage opportunities in this market are of the form

$$H = (0, -2H_2, H_2)^T, \qquad \text{with } H_2 < 0.$$

5. This a trivial exercise in linear algebra. Consider two arbitrary elements $\hat{X} = \sum_{n=1}^{N} \hat{H}_n \Delta S_n^*$ and $\tilde{X} = \sum_{n=1}^{N} \tilde{H}_n \Delta S_n^*$ of W and check that $a\hat{X} + b\tilde{X} \in W$. The orthogonal complement of a linear subspace is always a linear subspace.

6. The solution is

   (a)

$$W = \left\{ X = (X_1, X_2)^T \in \mathbb{R}^2 : X_1 + X_2 = 0 \right\},$$
$$W^\perp = \left\{ Y = (Y_1, Y_2)^T \in \mathbb{R}^2 : Y_1 - Y_2 = 0 \right\}.$$

   (b)

$$W = \left\{ X = (X_1, X_2, X_3)^T \in \mathbb{R}^3 : X_1 + X_2 = 0, 2X_1 + X_3 = 0 \right\},$$
$$W^\perp = \left\{ Y = (Y_1, Y_2, Y_3)^T \in \mathbb{R}^3 : Y_1 - Y_2 - 2Y_3 = 0 \right\}.$$

   (c)

$$W = \left\{ X = (X_1, X_2, X_3, X_4)^T \in \mathbb{R}^4 : X_1 + X_3 = 0, 2X_1 + 2X_2 + X_4 = 0 \right\},$$
$$W^\perp = \left\{ Y = (Y_1, Y_2, Y_3, Y_4)^T \in \mathbb{R}^4 : Y_2 - 2Y_4 = 0, 2Y_3 - 2Y_1 + Y_2 = 0 \right\}.$$

7. We need to solve the equation

$$Q^T S^* (1, \Omega) = (1, S_1^* (0), S_2^* (0)),$$

   for $Q \in \mathbb{R}^4$. The solutions can be parametrized as follows

$$Q = \left(\frac{1}{2} (1 - \lambda), \lambda, \frac{1}{2} - \lambda, \frac{\lambda}{2} \right)^T \qquad \text{with } \lambda \in \left(0, \frac{1}{2} \right).$$

   Hence, by the first fundamental theorem of asset pricing we can conclude that the market is arbitrage free.

8. For $d < 1 + r < u$ there exists a unique risk neutral measure given by

$$Q = \left(\frac{1 + r - d}{u - d}, \frac{u - (1 + r)}{u - d} \right)^T.$$

   If $u \leq 1+r$, the trading strategies $H = (-H_1, H_1)$ with $H_1 < 0$ are all the arbitrage opportunities. If $d \geq 1 + r$, the trading strategies $H = (-H_1, H_1)$ with $H_1 > 0$ are all the arbitrage opportunities.

9. The price of the put option is $\frac{1}{4}$ and the replicating trading strategy is $H = \left(\frac{3}{2}, -\frac{1}{4}\right)^T$.

10. Note first the identity

$$(K - S(1))^+ = (S(1) - K)^+ + K - S(1).$$

The call option is attainable iff there exists $H = (H_0, H_1)^T$ such that $(S(1) - K)^+ = H_0(1+r) + H_1 S(1)$. Using the previous identity we have that the put option is also attainable with the trading strategy $\tilde{H} = \left(H_0 + \frac{K}{1+r}, H_1 - 1\right)^T$. Moreover, note that $C = H_0 + H_1 S(0)$ and $P = \tilde{H}_0 + \tilde{H}_1 S(0) = H_0 + \frac{K}{1+r} + (H_1 - 1) S(0)$, from which the put-call parity follows.

11. From exercise 7 we know that this market has infinitely many risk neutral measures. By the second fundamental theorem of asset pricing this is equivalent to the market being incomplete. The set of attainable claims is given by

$$\left\{X = (X_1, X_2, X_3, X_4)^T \in \mathbb{R}^4 : -X_1 + 2X_2 - 2X_3 + X_4 = 0\right\}.$$

$X = (40, 30, 20, 10)^T$ does not belong to the previous set and, hence, it is not attainable. The arbitrage free interval for the price of X is $[V_-(X), V_+(X)] = [24.75, 27]$.

12. The call option is attainable for $K \geq 20/3$ or $K \leq 30/9$. According to exercise 10. the put option is attainable for the same values of K.

13. It follows from Steinitz exchange lemma. In particular, from the fact that given a set of linearly independent vectors of a linear subspace we can always extend the set to obtain a basis for the linear subspace. As $M = W^\perp \cap P^+ \neq \emptyset$, we choose $0 \neq Q_1 \in M$, which is linearly independent set because it is just one vector. Hence, we can extend $\{Q_1\}$ to a basis $\{Q_1, Y_2, ..., Y_J\}$ of $W^\perp$. Now, as $Q_1 > 0$, we can choose $a_2 \neq 0$ and $b_2 > 0$ such that $Q_2 := b_2(Q_1 + a_2 Y_2)$ is strictly positive and its components add up to one, i.e., $Q_2 \in M$. It is easy to check that $\{Q_1, Q_2\}$ is a set of linearly independent vectors. Hence, we can extend $\{Q_1, Q_2\}$ to a basis $\{Q_1, Q_2, Y_3, ..., Y_J\}$ of $W^\perp$. Obviously, we can repeat this procedure until we get a basis with all its elements in M. From exercise 7 we know that

$$M = \left\{Q_\lambda = \left(\frac{1}{2}(1-\lambda), \lambda, \frac{1}{2} - \lambda, \frac{\lambda}{2}\right)^T, \lambda \in \left(0, \frac{1}{2}\right)\right\}.$$

Moreover, from exercise 6 (c) it follows that $\dim\left(W^\perp\right) = 2$. Choosing, for instance, $Q_{1/3} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)^T$ and $Q_{1/4} = \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right)^T$ we get two linearly independent vectors in M.

14. The terminal value for any strategy $H = (H_0, H_1)^T$ satisfying $v = V(0) = H_0 + 5H_1$ is $V(1) = \left(\frac{10}{9}(v + H_1), \frac{10}{9}(v - H_1)\right)^T$:

(a) The optimal strategy is $H = ((6-10p)\,v, (2p-1)\,v)^T$ and the optimal expected utility is

$$\mathbb{E}\left[U\left(V\left(1\right)\right)\right] = p\log\left(\frac{20}{9}pv\right) + (1-p)\log\left(\frac{20}{9}(1-p)v\right).$$

(b) The optimal strategy is

$$H = \left(v - \frac{9}{4}\log\left(\frac{p}{1-p}\right), \frac{9}{20}\log\left(\frac{p}{1-p}\right)\right)^T,$$

and the optimal expected utility is

$$\mathbb{E}\left[U\left(V\left(1\right)\right)\right] = -2e^{-\frac{10}{9}v}p^{1/2}\left(1-p\right)^{1/2}.$$

(c) Let $A = \left(\frac{p}{1-p}\right)^{\frac{1}{\gamma-1}}$. Then, the optimal strategy is

$$H = \left(v\left(\frac{6A-4}{1+A}\right), v\frac{1-A}{1+A}\right)^T,$$

and the optimal expected utility is

$$\mathbb{E}\left[U\left(V\left(1\right)\right)\right] = \gamma^{-1}\left(\frac{20}{9}\right)^{\gamma}\left\{p\left(\frac{v}{1+A}\right)^{\gamma} + (1-p)\left(\frac{v}{1+A^{-1}}\right)^{\gamma}\right\}.$$

15. The optimal attainable wealth is $\hat{W} = \left(\frac{4}{3}v, \frac{8}{9}v\right)^T$, the optimal trading strategy is $H = \left(0, \frac{1}{5}v\right)^T$ and the optimal expected utility is

$$\mathbb{E}\left[U\left(\hat{W}\right)\right] = \log\left(v\right) - \frac{3}{5}\log\left(\frac{3}{4}\right) - \frac{2}{5}\log\left(\frac{9}{8}\right).$$

16. Let $\Gamma_1 := \frac{3}{5}\left(\frac{3}{4}\right)^{-\frac{\gamma}{1-\gamma}} + \frac{2}{5}\left(\frac{9}{8}\right)^{-\frac{\gamma}{1-\gamma}}$ and $\Gamma_2 := \left(\frac{3}{4}\right)^{-\frac{1}{1-\gamma}} - \left(\frac{9}{8}\right)^{-\frac{1}{1-\gamma}}$. The optimal attainable wealth is

$$\hat{W} = \left(\frac{v}{\Gamma_1}\left(\frac{3}{4}\right)^{-\frac{1}{1-\gamma}}, \frac{v}{\Gamma_1}\left(\frac{9}{8}\right)^{-\frac{1}{1-\gamma}}\right)^T,$$

the optimal trading strategy is

$$H = \left(v\frac{4\Gamma_1 - 9\Gamma_2}{4\Gamma_1}, v\frac{9}{20}\frac{\Gamma_2}{\Gamma_1}\right)^T$$

and the optimal expected utility is

$$\mathbb{E}\left[U\left(\hat{W}\right)\right] = \frac{1}{\gamma}\left(\frac{v}{\Gamma_1}\right)^{\gamma}\left(\frac{3}{5}\left(\frac{3}{4}\right)^{-\frac{\gamma}{1-\gamma}} + \frac{2}{5}\left(\frac{9}{8}\right)^{-\frac{\gamma}{1-\gamma}}\right).$$