An Introduction to Mathematical Finance | Exercises 4
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Multiple Period Financial Markets

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1. M is a martingale if and only if M is an $\mathbb{F}$-adapted process satisfying

$$\mathbb{E}\left[M(t+s) \middle| \mathcal{F}_t \right] = M(t), \qquad t, s \geq 0. \tag{1}$$

Prove that is equivalent to

$$\mathbb{E}\left[M(t+1) \middle| \mathcal{F}_t \right] = M(t), \qquad t = 0, ..., T-1. \tag{2}$$

(a) That (1) implies (2) follows from setting $s = 1$ in (1).

(b) That (2) implies (1) follows by applying several times the tower property of the conditional expectation, that is, if $\mathcal{G}_1 \subseteq \mathcal{G}_2$ then

$$\mathbb{E}\left[X \middle| \mathcal{G}_1 \right] = \mathbb{E}\left[\mathbb{E}\left[X \middle| \mathcal{G}_2 \right] \middle| \mathcal{G}_1 \right] = \mathbb{E}\left[\mathbb{E}\left[X \middle| \mathcal{G}_1 \right] \middle| \mathcal{G}_2 \right]. \tag{3}$$

Since $\mathbb{F}$ is a filtration wehave that $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, $t = 0, ..., T-1$. Fix $t \in \{0, ..., T\}$ and $s \in \{0, ..., T-t\}$, then

$$\begin{aligned}
\mathbb{E}\left[M(t+s) \middle| \mathcal{F}_t \right] &\overset{(3)}{=} \mathbb{E}\left[\mathbb{E}\left[M(t+s) \middle| \mathcal{F}_{t+s-1} \right] \middle| \mathcal{F}_t \right] \\
&\overset{(2)}{=} \mathbb{E}\left[M(t+s-1) \middle| \mathcal{F}_t \right] \\
&\overset{(3)}{=} \mathbb{E}\left[\mathbb{E}\left[M(t+s-1) \middle| \mathcal{F}_{t+s-2} \right] \middle| \mathcal{F}_t \right] \\
&\overset{(2)}{=} \mathbb{E}\left[M(t+s-2) \middle| \mathcal{F}_t \right] \\
&\vdots \\
&\overset{(3)}{=} \mathbb{E}\left[\mathbb{E}\left[M(t+2) \middle| \mathcal{F}_{t+1} \right] \middle| \mathcal{F}_t \right] \\
&\overset{(2)}{=} \mathbb{E}\left[M(t+1) \middle| \mathcal{F}_t \right] \\
&\overset{(2)}{=} M(t).
\end{aligned}$$

(c) To prove that a martingale has constant expectation with value $\mathbb{E}[M(0)]$ note that

$$\mathbb{E}[M(0)] = \mathbb{E}[\mathbb{E}[M(t)|\mathcal{F}_0]] = \mathbb{E}[M(t)],$$

where in the first equality we have used (1) and in the second equality we have used the law of total expectation ($\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$). Moreover, note that if $\mathcal{F}_0 = \{\emptyset, \Omega\}$, then $M(0)$ is a constant and $\mathbb{E}[M(0)] = M(0)$.

2. To prove that the process $M = \{M(t)\}_{t=0,\ldots,T}$ defined by

$$\begin{aligned} M(0) &= 0, \\ M(t) &= \sum_{u=1}^{t} H(u)(S(u) - S(u-1)), \end{aligned}$$

is a martingale first we have to prove that $M(t)$ is $\mathbb{F}$-adapted. First note that if X and Y are $\mathcal{G}$-measurable with respect to an algebra $\mathcal{G}$ on Ω, then XY and $X+Y$ are $\mathcal{G}$-measurable. The process H is predictable and, in particular, adapted to $\mathbb{F}$. The process S is adapted to $\mathbb{F}$ because it is an $\mathbb{F}$-martingale, moreover $S(u-1)$ is also $\mathcal{F}_u$-measurable because $\mathcal{F}_{u-1} \subseteq \mathcal{F}_u$. Therefore, $H(u)(S(u) - S(u-1))$ is $\mathcal{F}_u$-measurable for $u \leq t$. As $\mathbb{F}$ is a filtration, $\mathcal{F}_u \subseteq \mathcal{F}_t$, and we can conclude that $M(t)$ is $\mathcal{F}_t$-measurable and, hence, M is $\mathbb{F}$-adapted. To prove the martingale property, first note that

$$M(t+1) = M(t) + H(t+1)(S(t+1) - S(t)). \tag{4}$$

Then,

$$\begin{aligned} \mathbb{E}[M(t+1)|\mathcal{F}_t] &\overset{(a)}{=} \mathbb{E}[M(t) + H(t+1)(S(t+1) - S(t))|\mathcal{F}_t] \\ &\overset{(b)}{=} \mathbb{E}[M(t)|\mathcal{F}_t] + \mathbb{E}[H(t+1)(S(t+1) - S(t))|\mathcal{F}_t] \\ &\overset{(c)}{=} M(t) + H(t+1)\mathbb{E}[(S(t+1) - S(t))|\mathcal{F}_t] \\ &\overset{(d)}{=} M(t), \end{aligned}$$

where we have used: (a) Equation (4), (b) Linearity of the conditional expectation, (c) $H(t+1)$ is $\mathcal{F}_t$-measurable and it can factor out of $\mathbb{E}[\cdot|\mathcal{F}_t]$, (d) S is a martingale and, therefore,

$$\mathbb{E}[S(t+1)|\mathcal{F}_t] = S(t) \Longleftrightarrow \mathbb{E}[(S(t+1) - S(t))|\mathcal{F}_t] = 0.$$

3. Let X be a random variable on a finite probability space $(\Omega, \mathcal{F}, P)$, $\mathbb{F}$ a filtration on Ω. We have to prove that $Y = \{Y(t) = \mathbb{E}[X|\mathcal{F}_t]\}$ is a martingale. That Y is $\mathbb{F}$-adapted is trivial, because by definition $\mathbb{E}[X|\mathcal{F}_t]$ is $\mathcal{F}_t$-measurable, which implies that $Y(t)$ is $\mathcal{F}_t$-measurable. To show the martingale property we can write

$$\mathbb{E}[Y(t+1)|\mathcal{F}_t] \overset{(a)}{=} \mathbb{E}[\mathbb{E}[X|\mathcal{F}_{t+1}]|\mathcal{F}_t] \overset{(b)}{=} \mathbb{E}[X|\mathcal{F}_t] \overset{(a)}{=} Y(t), \qquad t = 0, ..., T-1,$$

where we have used: (a) Definition of Y, (b) That $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ and the tower law of conditional expectation.

4. Let X be a random variable on a finite probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{G}$ an algebra on Ω and $\varphi : \mathbb{R} \to \mathbb{R}$ a convex function.

   (a) We have to prove that

$$\varphi\left(\mathbb{E}\left[X|\,\mathcal{G}\right](\omega)\right) \leq \mathbb{E}\left[\varphi\left(X\right)|\,\mathcal{G}\right](\omega), \qquad \omega \in \Omega. \tag{5}$$

Note that if φ is convex then for any $0 \leq \alpha_i \leq 1, i = 1, ..., N$ such that $\sum_{i=1}^{N} \alpha_i = 1$ we have

$$\varphi\left(\sum_{i=1}^{N} \alpha_i x_i\right) \leq \sum_{i=1}^{N} \alpha_i \varphi\left(x_i\right). \tag{6}$$

Let $\{A_1, ..., A_m\}$ be the partition on Ω such that $\mathcal{G} = \mathfrak{a}\left(\{A_1, ..., A_m\}\right)$. Then,

$$\mathbb{E}\left[X|\,\mathcal{G}\right](\omega) = \sum_{i=1}^{m} \mathbb{E}\left[X|\,A_i\right] \mathbf{1}_{A_i}(\omega) = \sum_{i=1}^{m} \frac{\mathbb{E}\left[X\mathbf{1}_{A_i}\right]}{P\left(A_i\right)} \mathbf{1}_{A_i}(\omega), \tag{7}$$

and

$$\mathbb{E}\left[\varphi\left(X\right)|\,\mathcal{G}\right](\omega) = \sum_{i=1}^{m} \mathbb{E}\left[\varphi\left(X\right)|\,A_i\right] \mathbf{1}_{A_i}(\omega) = \sum_{i=1}^{m} \frac{\mathbb{E}\left[\varphi\left(X\right)\mathbf{1}_{A_i}\right]}{P\left(A_i\right)} \mathbf{1}_{A_i}(\omega). \tag{8}$$

To check (5) take an arbitrary $\hat{\omega} \in \Omega$ and let j be the unique index in $\{1, ..., m\}$ such that $\hat{\omega} \in A_j$, then

$$\begin{aligned}
\varphi\left(\mathbb{E}\left[X|\,\mathcal{G}\right](\hat{\omega})\right) &\overset{(a)}{=} \varphi\left(\sum_{i=1}^{m} \frac{\mathbb{E}\left[X\mathbf{1}_{A_i}\right]}{P\left(A_i\right)} \mathbf{1}_{A_i}(\hat{\omega})\right) \overset{(b)}{=} \varphi\left(\frac{\mathbb{E}\left[X\mathbf{1}_{A_j}\right]}{P\left(A_j\right)}\right) \\
&\overset{(c)}{=} \varphi\left(\sum_{\omega \in A_j} X(\omega) \frac{P(\omega)}{P\left(A_j\right)}\right) \overset{(d)}{\leq} \sum_{\omega \in A_j} \varphi\left(X(\omega)\right) \frac{P(\omega)}{P\left(A_j\right)} \\
&\overset{(e)}{=} \frac{\mathbb{E}\left[\varphi\left(X\right)\mathbf{1}_{A_j}\right]}{P\left(A_j\right)} \overset{(b)}{=} \sum_{i=1}^{m} \frac{\mathbb{E}\left[\varphi\left(X\right)\mathbf{1}_{A_i}\right]}{P\left(A_i\right)} \mathbf{1}_{A_i}(\hat{\omega}) \\
&\overset{(f)}{=} \mathbb{E}\left[\varphi\left(X\right)|\,\mathcal{G}\right](\hat{\omega}),
\end{aligned}$$

where we have used: (a) The expression in (7), (b) $\mathbf{1}_{A_i}(\hat{\omega}) \equiv 0$ if $i \neq j$ and $\mathbf{1}_{A_j}(\hat{\omega}) \equiv 1$, (c) We compute $\mathbb{E}\left[X\mathbf{1}_{A_j}\right]$, (d) We apply (6), taking into account that $0 < \frac{P(\omega)}{P(A_j)} < 1$ and $\sum_{\omega \in A_i} \frac{P(\omega)}{P(A_j)} = 1$, (e) We have that $\sum_{\omega \in A_j} \varphi\left(X(\omega)\right) P(\omega) = \mathbb{E}\left[\varphi\left(X\right)\mathbf{1}_{A_j}\right]$, (f) The expression in (8).

(b) We have to prove that if $Y = \{Y(t)\}_{t=0,...,T}$ is a $\mathbb{F}$-martingale then $Z = \{Z(t) = \varphi(Y(t))\}_{t=0,...,T}$ is a $\mathbb{F}$-submartingale. For every $t = 0, ..., T, Z(t)$ is $\mathfrak{a}(Y(t))$-measurable because $Z(t)$ is obtained applying a measurable function to $Y(t)$ (alternatively, $Z(t)$ is constant on the elements of the partition $\pi_{Y(t)}$ and, therefore, $\pi_{Z(t)} \subseteq \pi_{Y(t)}$ and $\mathfrak{a}(Z(t)) \subseteq \mathfrak{a}(Y(t))$). Since $Y(t)$ is $\mathcal{F}_t$-measurable, $\mathfrak{a}(Z(t)) \subseteq \mathfrak{a}(Y(t)) \subseteq \mathcal{F}_t$, $Z(t)$ is $\mathcal{F}_t$-measurable and Z is $\mathbb{F}$-adapted. To prove the submartingale property, note that

$$\begin{aligned} \mathbb{E}[Z(t+1)|\mathcal{F}_t] &\overset{(a)}{=} \mathbb{E}[\varphi(Y(t+1))|\mathcal{F}_t] \overset{(b)}{\geq} \varphi(\mathbb{E}[Y(t+1)|\mathcal{F}_t]) \\ &\overset{(c)}{=} \varphi(Y(t)) \overset{(a)}{=} Z(t), \end{aligned}$$

where we have used: (a) Definition of Z, (b) The property of conditional expectation proved in the previous section, (c) Y is a martingale.

5. Let $(\Omega, \mathcal{F}, P)$ be a finite probability space , $\mathbb{F}$ a filtration on Ω such that $\mathcal{F} = \mathcal{F}_T$, Q a probability measure on Ω such that $Q > 0$ and L the stochastic process defined by

$$L = \left\{ L(t) = \mathbb{E}\left[\frac{Q}{P} \middle| \mathcal{F}_t \right] \right\}_{t=0,...,T}$$

(a) We have to show that $L > 0$ and $L(0) = 1$. We will first show that if X is a strictly positive random variable and $\mathcal{G}$ is an algebra on Ω, then $\mathbb{E}[X|\mathcal{G}]$ is a strictly positive random variable. Let $A := \{\omega \in \Omega : \mathbb{E}[X|\mathcal{G}](\omega) = 0\}$. We want to show that $A = \emptyset$. Suppose that $A \neq \emptyset$. Note that $A \in \mathcal{G}$, because $\mathbb{E}[X|\mathcal{G}]$ is G-measurable and, therefore,

$$A = \mathbb{E}[X|\mathcal{G}]^{-1}(0) \in \mathfrak{a}(\mathbb{E}[X|\mathcal{G}]) \subseteq \mathcal{G}.$$

Moreover, $\mathbb{E}[X|\mathcal{G}]\mathbf{1}_A \equiv 0$ and, hence, $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_A] = 0$. But, on the other hand,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_A] \overset{(a)}{=} \mathbb{E}[X\mathbf{1}_A] \overset{(b)}{=} \sum_{\omega \in A} X(\omega) P(\omega) \overset{(c)}{>} 0,$$

where we have used: (a) Definition of conditional expectation and $A \in \mathcal{G}$, (b) Definition of expectation, (c) $X(\omega) > 0$ and $P(\omega) > 0$. Hence we have a contradiction and we can conclude that $A = \emptyset$ and $\mathbb{E}[X|\mathcal{G}](\omega) > 0, \omega \in \Omega$. Finally, using the results in Exercise 1, we get

$$L(0) = \mathbb{E}\left[\frac{Q}{P} \middle| \mathcal{F}_0 \right] = \mathbb{E}\left[\frac{Q}{P} \right] = E_Q[1] = 1.$$

(b) We have to prove the formula

$$\mathbb{E}_Q[W|\mathcal{F}_t] = \frac{\mathbb{E}[WL(T)|\mathcal{F}_t]}{L(t)}.$$

First note that, $\frac{Q}{P}$ is $\mathcal{F}$-measurable and, since $\mathcal{F} = \mathcal{F}_T$, $\frac{Q}{P}$ is $\mathcal{F}_T$-measurable, $L(T) = \frac{Q}{P}$. Then, note that $\mathbb{E}\left[\left. W L(T) \right| \mathcal{F}_t\right] L^{-1}(t)$ is $\mathcal{F}_t$-measurable because it is the quotient of two $\mathcal{F}_t$-measurable random variables and it is well defined because $L(t) > 0$. To prove the property defining the conditional expectation, let $A \in \mathcal{F}_t$, then

$$
\begin{aligned}
\mathbb{E}_Q\left[\frac{\mathbb{E}\left[\left. W L(T)\right| \mathcal{F}_t\right]}{L(t)} \mathbf{1}_A\right] &\overset{(a)}{=} \mathbb{E}\left[L(T) \frac{\mathbb{E}\left[\left. W L(T)\right| \mathcal{F}_t\right]}{L(t)} \mathbf{1}_A\right] \\
&\overset{(b)}{=} \mathbb{E}\left[\mathbb{E}\left[\left. L(T) \frac{\mathbb{E}\left[\left. W L(T)\right| \mathcal{F}_t\right]}{L(t)} \mathbf{1}_A\right| \mathcal{F}_t\right]\right] \\
&\overset{(c)}{=} \mathbb{E}\left[\mathbb{E}\left[\left. L(T)\right| \mathcal{F}_t\right] \frac{\mathbb{E}\left[\left. W L(T)\right| \mathcal{F}_t\right]}{L(t)} \mathbf{1}_A\right] \\
&\overset{(d)}{=} \mathbb{E}\left[\mathbb{E}\left[\left. W L(T)\right| \mathcal{F}_t\right] \mathbf{1}_A\right] \\
&\overset{(e)}{=} \mathbb{E}\left[W L(T) \mathbf{1}_A\right] \\
&\overset{(a)}{=} \mathbb{E}_Q\left[W \mathbf{1}_A\right],
\end{aligned}
$$

where we have used: (a) Definition of $\mathbb{E}_Q[\cdot]$ and $L(T) = \frac{Q}{P}$, (b) Law of total expectation, (c) $\mathbb{E}\left[\left. W L(T)\right| \mathcal{F}_t\right] L^{-1}(t) \mathbf{1}_A$ is $\mathcal{F}_t$-measurable and factors out of $\mathbb{E}\left[\left. \cdot\right| \mathcal{F}_t\right]$, (d) $L(t) = \mathbb{E}\left[\left. L(T)\right| \mathcal{F}_t\right]$, (e) $\mathbf{1}_A$ is $\mathcal{F}_t$-measurable and goes in $\mathbb{E}\left[\left. \cdot\right| \mathcal{F}_t\right]$ and then the law of total expectation.

(c) First we prove the implication $\Rightarrow$). We have that $Z = XL$ is $\mathbb{F}$-adapted because it is the product of two $\mathbb{F}$-adapted processes. Regarding the martingale condition, we have that

$$
\begin{aligned}
\mathbb{E}\left[\left. X(t+1) L(t+1)\right| \mathcal{F}_t\right] &\overset{(a)}{=} \mathbb{E}\left[\left. X(t+1) \mathbb{E}\left[\left. L(T)\right| \mathcal{F}_{t+1}\right]\right| \mathcal{F}_t\right] \\
&\overset{(b)}{=} \mathbb{E}\left[\left. \mathbb{E}\left[\left. X(t+1) L(T)\right| \mathcal{F}_{t+1}\right]\right| \mathcal{F}_t\right] \\
&\overset{(c)}{=} \mathbb{E}\left[\left. X(t+1) L(T)\right| \mathcal{F}_t\right] \\
&\overset{(d)}{=} \frac{\mathbb{E}\left[\left. X(t+1) L(T)\right| \mathcal{F}_t\right]}{L(t)} L(t) \\
&\overset{(e)}{=} \mathbb{E}_Q\left[\left. X(t+1)\right| \mathcal{F}_t\right] L(t) \\
&\overset{(f)}{=} X(t) L(t),
\end{aligned}
$$

where we have used: (a) Definition of the process L, (b) $X(t+1)$ is $\mathcal{F}_{t+1}$-measurable and goes in $\mathbb{E}\left[\left. \cdot\right| \mathcal{F}_{t+1}\right]$, (c) Tower law, (d) Divide and multiply by $L(t)$, (e) Formula for the conditional expectation under Q, (f) X is a martingale under Q.

Next we prove the implication $\Leftarrow$). We have that $X = Z/L$ is $\mathbb{F}$-adapted because it is the quotient of two $\mathbb{F}$-adapted processes with strictly positive denominator. Regarding the martingale condition, we have that

$$\begin{aligned}\mathbb{E}_Q\left[X(t+1)\middle|\mathcal{F}_t\right] &\overset{(a)}{=} \frac{\mathbb{E}\left[X(t+1)L(T)\middle|\mathcal{F}_t\right]}{L(t)}\\ &\overset{(b)}{=} \frac{\mathbb{E}\left[X(t+1)\mathbb{E}\left[L(T)\middle|\mathcal{F}_{t+1}\right]\middle|\mathcal{F}_t\right]}{L(t)}\\ &\overset{(c)}{=} \frac{\mathbb{E}\left[X(t+1)L(t+1)\middle|\mathcal{F}_t\right]}{L(t)}\\ &\overset{(d)}{=} \frac{X(t)L(t)}{L(t)} = X(t),\end{aligned}$$

where we have used: (a) Formula for the conditional expectation under Q, (b) Tower law and $X(t+1)$ is $\mathcal{F}_{t+1}$-measurable and goes out $\mathbb{E}\left[\cdot\middle|\mathcal{F}_{t+1}\right]$, (c) Definition of the process L, $(d) XL$ is a martingale under P.

6. We have a multiperiod market with $T=2, K=5, r=0, N=1, S(0)=6, S(1)=(5,5,5,7,7)^T$ and $S(2)=(3,4,8,6,8)^T$. We first compute the partitions associated to $S(0), S(1)$ and $S(2)$. We have

$$\begin{aligned}&\pi_{S(0)}=\{S(0)=6\}=\{\Omega\},\\ &\pi_{S(1)}=\{\{S(1)=5\},\{S(1)=7\}\}=\{\{\omega_1,\omega_2,\omega_3\},\{\omega_4,\omega_5\}\}=:\{A_{1,1},A_{1,2}\},\\ &\pi_{S(2)}=\{\{S(2)=3\},\{S(2)=4\},\{S(2)=6\},\{S(2)=8\}\}=\{\{\omega_1\},\{\omega_2\},\{\omega_5\},\{\omega_3,\omega_4\}\}\\ &\qquad=:\{A_{2,1},A_{2,2},A_{2,3},A_{2,4}\}.\end{aligned}$$

The partitions associated to $(S(0),S(1))$ and to $(S(0),S(1),S(2))$ are given by

$$\begin{aligned}\pi_{(S(0),S(1))} &= \pi_{S(0)}\cap\pi_{S(1)}=\{\Omega\cap A_{1,1},\Omega\cap A_{1,1}\}=\{A_{1,1},A_{1,2}\},\\ \pi_{(S(0),S(1),S(2))} &= \pi_{S(0)}\cap\pi_{S(1)}\cap\pi_{S(2)}=\pi_{S(0),S(1)}\cap\pi_{S(2)}\\ &=\{A_{1,1}\cap A_{2,1},A_{1,1}\cap A_{2,2},A_{1,1}\cap A_{2,3},A_{1,1}\cap A_{2,4}\\ &\qquad ,A_{1,2}\cap A_{2,1},A_{1,2}\cap A_{2,2},A_{1,2}\cap A_{2,3},A_{1,2}\cap A_{2,4}\}\\ &=\{\{\omega_1\},\{\omega_2\},\emptyset,\{\omega_3\},\emptyset,\emptyset,\{\omega_5\},\{\omega_4\}\}\\ &=\{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\},\{\omega_5\}\}.\end{aligned}$$

The filtrations are given by

$$\begin{aligned}&\mathcal{F}_0=\mathfrak{a}(S_1(0))=\mathfrak{a}(\{\Omega\})=\{\emptyset,\Omega\},\\ &\mathcal{F}_1=\mathfrak{a}(S_1(0),S_1(1))=\mathfrak{a}(\{A_{1,1},A_{1,2}\})=\{\emptyset,\Omega,A_{1,1},A_{1,2}\}=\{\emptyset,\Omega,\{\omega_1,\omega_2,\omega_3\},\{\omega_4,\omega_5\}\},\\ &\mathcal{F}_2=\mathfrak{a}(S_1(0),S_1(1),S_1(2))=\mathfrak{a}(\{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\},\{\omega_5\}\})=\mathcal{P}(\Omega),\end{aligned}$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω. Since $r=0$, we have that $S^*(t)=S(t), t=0,1,2$. We have to find the set of probability measures Q such that $S^*=\{S^*(t)\}_{t=0,1,2}$ is a martingale under Q.

- For $t = 1$, we have that

$$S(0) = S^*(0) = \mathbb{E}_Q[S^*(1)|\mathcal{F}_0] = \mathbb{E}_Q[S^*(1)] = \mathbb{E}_Q[S(1)],$$

if and only if

$$6 = 5(Q_1 + Q_2 + Q_3) + 7(Q_4 + Q_5).$$

- For $t = 2$, we have that

$$\begin{aligned} 5\mathbf{1}_{A_{1,1}} + 7\mathbf{1}_{A_{1,2}} &= S(1) = S^*(1) = \mathbb{E}_Q[S^*(2)|\mathcal{F}_1] = \mathbb{E}_Q[S(2)|\mathcal{F}_1] \\ &= \mathbb{E}_Q[S(2)|A_{1,1}]\mathbf{1}_{A_{1,1}} + \mathbb{E}_Q[S(2)|A_{1,2}]\mathbf{1}_{A_{1,2}}, \end{aligned}$$

where

$$\mathbb{E}_Q[S(2)|A_{1,1}] = \frac{\mathbb{E}_Q\left[S(2)\mathbf{1}_{A_{1,1}}\right]}{\mathbb{E}_Q\left[\mathbf{1}_{A_{1,1}}\right]} = \frac{3Q_1 + 4Q_2 + 8Q_3}{Q_1 + Q_2 + Q_3},$$

and

$$\mathbb{E}_Q[S(2)|A_{1,2}] = \frac{\mathbb{E}_Q\left[S(2)\mathbf{1}_{A_{1,2}}\right]}{\mathbb{E}_Q\left[\mathbf{1}_{A_{1,2}}\right]} = \frac{6Q_4 + 8Q_5}{Q_4 + Q_5}.$$

Then we get the equations

$$\begin{aligned} 5 &= \frac{3Q_1 + 4Q_2 + 8Q_3}{Q_1 + Q_2 + Q_3}, \\ 7 &= \frac{6Q_4 + 8Q_5}{Q_4 + Q_5}. \end{aligned}$$

Combining the previous equations with the fact that Q must be a probability measure we obtain the following set of equations

$$6 = 5(Q_1 + Q_2 + Q_3) + 7(Q_4 + Q_5), \tag{9}$$
$$0 = 2Q_1 + Q_2 - 3Q_3, \tag{10}$$
$$0 = Q_4 - Q_5, \tag{11}$$
$$1 = Q_1 + Q_2 + Q_3 + Q_4 + Q_5. \tag{12}$$

From (11) we get $Q_4 = Q_5$. Combining (11) and (12) we obtain

$$Q_4 + Q_5 = 1 - Q_1 - Q_2 - Q_3, \tag{13}$$

and pluggin the previous expression for $Q_4 + Q_5$ to (9) we get

$$1 = 2Q_1 + 2Q_2 + 2Q_3. \tag{14}$$

From (10) we have that $Q_2 = -2Q_1 + 3Q_3$ and plugging this expression in (14) gives

$$1 = -2Q_1 + 8Q_3 \Leftrightarrow Q_3 = \frac{1 + 2Q_1}{8}.$$

Hence,

$$Q_2 = -2Q_1 + 3\left(\frac{1+2Q_1}{8}\right) = \frac{3-10Q_1}{8}$$

and using (13) and we get

$$\begin{aligned} Q_4 + Q_5 &= 1 - Q_1 - \frac{3-10Q_1}{8} - \frac{1+2Q_1}{8} \\ &= \frac{8-3-1-Q_1(8-10+2)}{8} = \frac{1}{2}, \end{aligned}$$

which combined with $Q_4 = Q_5$ yields $Q_4 = Q_5 = \frac{1}{4}$. Finally, imposing that $Q_1 > 0, Q_2 > 0$ we get

$$Q_2 > 0 \Longleftrightarrow Q_1 < \frac{3}{10},$$

and the set of all martingale measures M is given by

$$\begin{aligned} M &= \left\{Q = \left(Q_1, \frac{3-10Q_1}{8}, \frac{1+2Q_1}{8}, \frac{1}{4}, \frac{1}{4}\right), 0 < Q_1 < \frac{3}{10}\right\} \\ &\overset{\lambda=2Q_1}{=} \left\{Q_\lambda = \left(\frac{\lambda}{2}, \frac{3-5\lambda}{8}, \frac{1+\lambda}{8}, \frac{1}{4}, \frac{1}{4}\right), 0 < \lambda < \frac{3}{5}\right\}. \end{aligned}$$

Let X be an arbitrary contingent claim. Since $M \neq \emptyset$, we know that X is attainable if and only if $\mathbb{E}_Q[X/B(2)]$ is constant with respect to $Q \in M$. Then,

$$\begin{aligned} \mathbb{E}_Q[X/B(2)] = \mathbb{E}_Q[X] &= X_1\frac{\lambda}{2} + X_2\frac{3-5\lambda}{8} + X_3\frac{1+\lambda}{8} + X_4\frac{1}{4} + X_5\frac{1}{4} \\ &= \lambda\left(\frac{X_1}{2} - \frac{5X_2}{8} + \frac{X_3}{8}\right) + \frac{3X_2}{8} + \frac{X_3}{8} + \frac{1}{4}(X_4 + X_5). \end{aligned}$$

The previous expression does not depend on λ if and only if

$$\frac{X_1}{2} - \frac{5X_2}{8} + \frac{X_3}{8} = 0 \Longleftrightarrow 4X_1 - 5X_2 + X_3 = 0.$$

The claim $X = (2,1,1,2,3)^T$ is not attainable because

$$4X_1 - 5X_2 + X_3 = 4 \times 2 - 5 \times 1 + 1 = 4 \neq 0.$$

Hence, there is an interval of arbitrage free prices $[V_-(X), V_+(X)]$, where $V_-(X)$ is the lower hedging price of X and $V_+(X)$ is the upper hedging price of X. Moreover, we know that

$$V_-(X) = \inf_{Q \in M}\left\{\mathbb{E}_Q\left[\frac{X}{B(2)}\right]\right\} = \inf_{\lambda \in \left(0, \frac{3}{5}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(2)}\right]\right\},$$

and

$$V_{+}(X)=\sup_{Q\in M}\left\{\mathbb{E}_{Q}\left[\frac{X}{B(2)}\right]\right\}=\sup_{\lambda\in\left(0,\frac{3}{5}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(2)}\right]\right\}.$$

We have that

$$\begin{aligned}\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(2)}\right]&=\mathbb{E}_{Q_\lambda}[X]=\lambda\left(\frac{2}{2}-\frac{5}{8}+\frac{1}{8}\right)+\frac{3}{8}+\frac{1}{8}+\frac{1}{4}(2+3)\\&=\frac{\lambda}{2}+\frac{7}{4}.\end{aligned}$$

The previous computation yields

$$\begin{aligned}V_{-}(X)&=\inf_{\lambda\in\left(0,\frac{3}{5}\right)}\left\{\frac{\lambda}{2}+\frac{7}{4}\right\}=\frac{0}{2}+\frac{7}{4}=\frac{7}{4},\\V_{+}(X)&=\sup_{\lambda\in\left(0,\frac{3}{5}\right)}\left\{\frac{\lambda}{2}+\frac{7}{4}\right\}=\frac{1}{2}\frac{3}{5}+\frac{7}{4}=\frac{41}{20}.\end{aligned}$$

7. Consider a 2-period market with $\Omega=\{\omega_1,...,\omega_4\}$, $P=(1/4,1/4,1/4,1/4)^T$, $r=0$, and one risky security with $S(0)=5$,

$$S(1)=(8,8,4,4)^T,\quad S(2)=(9,6,6,3)^T.$$

We have to compute the optimal attainable wealth, the optimal objective value, and the optimal trading strategy under the utility function $U(u)=-u^{-1}$. We first compute the partitions associated to $S(0)$, $S(1)$ and $S(2)$. We have

$$\begin{aligned}&\pi_{S(0)}=\{S(0)=5\}=\{\Omega\},\\&\pi_{S(1)}=\{\{S(1)=4\},\{S(1)=8\}\}=\{\{\omega_3,\omega_4\},\{\omega_1,\omega_2\}\}=:\{A_{1,1},A_{1,2}\},\\&\pi_{S(2)}=\{\{S(2)=3\},\{S(2)=6\},\{S(2)=9\}\}=\{\{\omega_4\},\{\omega_2,\omega_3\},\{\omega_1\}\}=:\{A_{2,1},A_{2,2},A_{2,3}\}.\end{aligned}$$

The partitions associated to $(S(0),S(1))$ and to $(S(0),S(1),S(2))$ are given by

$$\begin{aligned}\pi_{(S(0),S(1))}&=\pi_{S(0)}\cap\pi_{S(1)}=\{\Omega\cap A_{1,1},\Omega\cap A_{1,1}\}=\{A_{1,1},A_{1,2}\},\\\pi_{(S(0),S(1),S(2))}&=\pi_{S(0)}\cap\pi_{S(1)}\cap\pi_{S(2)}=\pi_{S(0),S(1)}\cap\pi_{S(2)}\\&=\{A_{1,1}\cap A_{2,1},A_{1,1}\cap A_{2,2},A_{1,1}\cap A_{2,3},A_{1,2}\cap A_{2,1},A_{1,2}\cap A_{2,2},A_{1,2}\cap A_{2,3}\}\\&=\{\{\omega_4\},\{\omega_3\},\emptyset,\emptyset,\{\omega_2\},\{\omega_1\}\}=\{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\}.\end{aligned}$$

The filtrations are given by

$$\begin{aligned}\mathcal{F}_0&=\mathfrak{a}(S(0))=\mathfrak{a}(\{\Omega\})=\{\emptyset,\Omega\},\\\mathcal{F}_1&=\mathfrak{a}(S(0),S(1))=\mathfrak{a}(\{A_{1,1},A_{1,2}\})=\{\emptyset,\Omega,A_{1,1},A_{1,2}\},\\\mathcal{F}_2&=\mathfrak{a}(S(0),S(1),S(2))=\mathfrak{a}(\{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\})=\mathcal{P}(\Omega),\end{aligned}$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω.

This market is the same as Example 7.12 seen in class. Hence, we know that, given r, there exist a unique martingale measure Q given by

$$Q = \left(\frac{(1+5r)(2+8r)}{12}, \frac{(1+5r)(1-8r)}{12}, \frac{(3-5r)(1+4r)}{12}, \frac{(3-5r)(2-4r)}{12} \right)^T.$$

As in this exercise $r = 0$, we get that

$$Q = \left(\frac{1}{6}, \frac{1}{12}, \frac{1}{4}, \frac{1}{2} \right)^T.$$

Since $M = \{Q\}$ the market is arbitrage free and complete, due to the first and second fundamental theorem of asset pricing. Then, we can use the martingale method to solve the optimal portfolio problem. In this setup, $M = \{Q\}$, the martingale method consists in the following two steps:

1. We first solve the constrained optimization problem

$$\max_W \mathbb{E}\left[U(W)\right]$$
$$\text{subject to } \mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v,$$

   and obtain the optimal attainable wealth $\widehat{W}$.

2. Given $\widehat{W}$, we find the optimal trading strategy $\widehat{H}$ such that its associated value process $\widehat{V}$ replicates $\widehat{W}$, that is, $\widehat{V}(2) = \widehat{W}$.

The previous constrained problem can be solved using the Lagrange multipliers method. The optimal attainable wealth $\widehat{W}$ is given by

$$\widehat{W} = I\left(\frac{\widehat{\lambda} L}{B(2)}\right),$$

where I is the inverse of $U'(u)$, L is the state-price density vector $L = \frac{Q}{P}$, $B(2)$ is the price of the risk-less asset at time 2 and $\widehat{\lambda}$ is the optimal Lagrange multiplier associated to the constraint $\mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v$. Taking into account that $r = 0$,$U(u) = -u^{-1}$, $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$ and $Q = \left(\frac{1}{6}, \frac{1}{12}, \frac{1}{4}, \frac{1}{2}\right)^T$,we have that

$$i = U'(u) = u^{-2} \iff I(i) = u = i^{-1/2},$$
$$L = \left(\frac{\frac{1}{6}}{\frac{1}{4}}, \frac{\frac{1}{12}}{\frac{1}{4}}, \frac{\frac{1}{4}}{\frac{1}{4}}, \frac{\frac{1}{2}}{\frac{1}{4}} \right)^T = \left(\frac{2}{3}, \frac{1}{3}, 1, 2 \right)^T,$$
$$B(2) = 1,$$

which yield $\widehat{W} = \left(\widehat{\lambda} L\right)^{-1/2}$. The optimal Lagrange multiplier $\widehat{\lambda}$ satisfies the equation

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B(2)}\right] = \mathbb{E}_Q\left[\frac{I\left(\frac{\widehat{\lambda} L}{B(2)}\right)}{B(2)}\right] = \mathbb{E}_Q\left[\left(\widehat{\lambda} L\right)^{-1/2}\right] = \left(\widehat{\lambda}\right)^{-1/2} \mathbb{E}_Q\left[L^{-1/2}\right].$$

Therefore, we get

$$\widehat{\lambda} = \left(\frac{\mathbb{E}_Q\left[L^{-1/2}\right]}{v}\right)^2, \qquad \widehat{W} = v \frac{L^{-1/2}}{\mathbb{E}_Q\left[L^{-1/2}\right]},$$

and the optimal expected utility is given by

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right] = \mathbb{E}\left[-\left(v \frac{L^{-1/2}}{\mathbb{E}_Q\left[L^{-1/2}\right]}\right)^{-1}\right] = -v^{-1} \frac{\mathbb{E}\left[L^{1/2}\right]}{\mathbb{E}_Q\left[L^{-1/2}\right]^{-1}} = -v^{-1}\left(\mathbb{E}_Q\left[L^{-1/2}\right]\right)^2,$$

where we have used that $\mathbb{E}\left[L^{1/2}\right] = \mathbb{E}\left[L L^{-1/2}\right] = \mathbb{E}_Q\left[L^{-1/2}\right]$. Next we need to compute $L^{-1/2}$ and $\mathbb{E}_Q\left[L^{-1/2}\right]$. We have that

$$\begin{aligned} L^{-1/2} &= \left(\left(\frac{2}{3}\right)^{-1/2}, \left(\frac{1}{3}\right)^{-1/2}, (1)^{-1/2}, (2)^{-1/2}\right)^T \\ &= \left(\sqrt{\frac{3}{2}}, \sqrt{3}, 1, \sqrt{\frac{1}{2}}\right)^T, \end{aligned}$$

and

$$\mathbb{E}_Q\left[L^{-1/2}\right] = \sqrt{\frac{3}{2}}\frac{1}{6} + \sqrt{3}\frac{1}{12} + 1\frac{1}{4} + \sqrt{\frac{1}{2}}\frac{1}{2} = \frac{1}{12}\left(1+\sqrt{2}\right)\left(3+\sqrt{3}\right),$$

which yield

$$\begin{aligned} \mathbb{E}\left[U\left(\widehat{W}\right)\right] &= -v^{-1}\left(\frac{1}{12}\left(1+\sqrt{2}\right)\left(3+\sqrt{3}\right)\right)^2 \\ &= -\frac{1}{24v}\left(3+2\sqrt{2}\right)\left(2+\sqrt{3}\right). \end{aligned}$$

and

$$\widehat{W} = \begin{pmatrix} \frac{12v}{\left(1+\sqrt{2}\right)\left(3+\sqrt{3}\right)}\sqrt{\frac{3}{2}} \\ \frac{12v}{\left(1+\sqrt{2}\right)\left(3+\sqrt{3}\right)}\sqrt{3} \\ \frac{12v}{\left(1+\sqrt{2}\right)\left(3+\sqrt{3}\right)}1 \\ \frac{12v}{\left(1+\sqrt{2}\right)\left(3+\sqrt{3}\right)}\sqrt{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} v3\left(2-\sqrt{2}\right)\left(\sqrt{3}-1\right) \\ v6\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right) \\ v2\left(\sqrt{2}-1\right)\left(3-\sqrt{3}\right) \\ v6\left(2-\sqrt{2}\right)\left(3+\sqrt{3}\right)^{-1} \end{pmatrix}.$$

Finally, we have to compute the optimal trading strategy $\widehat{H} = \left\{ \left(\widehat{H}_0(t), \widehat{H}_1(t) \right)^T \right\}_{t=1,2}$, that is, a self-financing and predictable process such that its asociated value process $\widehat{V}$ satisfies $\widehat{V}(2) = \widehat{W}$. We first compute the discounted increments of the risky asset

$$\begin{aligned}\Delta S^*(2) &= \Delta S(2) = (1, -2, 2, -1)^T, \\ \Delta S^*(1) &= \Delta S(1) = (3, 3, -1, -1)^T.\end{aligned}$$

- For $t = 2$, using that $\widehat{H}$ must be self-financing we have that $\widehat{W} = \frac{\widehat{W}}{B(2)} = \widehat{W^*} = \widehat{V}^*(1) + \widehat{H}_1(2)\Delta S^*(2)$.
  - Assuming that $\omega \in A_{1,1} = \{\omega_3, \omega_4\}$ and the predictability of $\widehat{H}$ we get the equations

    $$\begin{aligned} v2\left(\sqrt{2}-1\right)\left(3-\sqrt{3}\right) &= \widehat{W}_3 = \widehat{V}^*(1,\omega_3) + \widehat{H}_1(2,\omega_3) \times 2, \\ v6\left(2-\sqrt{2}\right)\left(3+\sqrt{3}\right)^{-1} &= \widehat{W}_4 = \widehat{V}^*(1,\omega_4) + \widehat{H}_1(2,\omega_4) \times (-1), \\ \widehat{V}^*(1,\omega_3) &= \widehat{V}^*(1,\omega_4), \\ \widehat{H}_1(2,\omega_3) &= \widehat{H}_1(2,\omega_4), \end{aligned}$$

    which, using that $r = 0$, yield

    $$\begin{aligned} \widehat{V}^*(1,\omega_3) = \widehat{V}^*(1,\omega_4) = \widehat{V}(1,\omega_3) = \widehat{V}(1,\omega_4) &= 4\left(3+\sqrt{3}\right)^{-1} v, \\ \widehat{H}_1(2,\omega_3) = \widehat{H}_1(2,\omega_4) &= \left(6\sqrt{2}-8\right)\left(3+\sqrt{3}\right)^{-1} v. \end{aligned}$$

  - Assuming that $\omega \in A_{1,2} = \{\omega_1, \omega_2\}$ and the predictability of $\widehat{H}$ we get the equations

    $$\begin{aligned} v3\left(2-\sqrt{2}\right)\left(\sqrt{3}-1\right) &= \widehat{W}_1 = \widehat{V}^*(1,\omega_1) + \widehat{H}_1(2,\omega_1) \times 1, \\ v6\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right) &= \widehat{W}_2 = \widehat{V}^*(1,\omega_2) + \widehat{H}_1(2,\omega_2) \times (-2), \\ \widehat{V}^*(1,\omega_1) &= \widehat{V}^*(1,\omega_2), \\ \widehat{H}_1(2,\omega_1) &= \widehat{H}_1(2,\omega_2), \end{aligned}$$

    which, using that $r = 0$, yield

    $$\begin{aligned} \widehat{V}^*(1,\omega_1) = \widehat{V}^*(1,\omega_2) = \widehat{V}(1,\omega_1) = \widehat{V}(1,\omega_2) &= 2\left(\sqrt{3}-1\right) v, \\ \widehat{H}_1(2,\omega_1) = \widehat{H}_1(2,\omega_2) &= \left(4-3\sqrt{2}\right)\left(\sqrt{3}-1\right) v. \end{aligned}$$

- For $t = 1$, the predictability assumption implies that $\widehat{H}_1(1)$ is constant. Moreover, using that $\widehat{H}$ must be self-financing we have that $\widehat{V}^*(1) = \widehat{V}^*(0) + \widehat{H}_1(1)\,\Delta S^*(1)$ and we get the following two equations

$$\begin{aligned} 4\left(3+\sqrt{3}\right)^{-1} v &= \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (-1), \qquad (\text{for } \omega \in A_{1,1}) \\ 2\left(\sqrt{3}-1\right) v &= \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (3), \qquad (\text{for } \omega \in A_{1,2}) \end{aligned}$$

which, using that $r = 0$, yield

$$\widehat{V}^*(0) = \widehat{V}(0) = v, \qquad \widehat{H}_1(1) = \frac{2-\sqrt{3}}{\sqrt{3}} v.$$

- Finally we compute $\widehat{H}_0(1)$ and $\widehat{H}_0(2)$ from the definition of value process. We have

$$\begin{aligned} \widehat{H}_0(1) = \widehat{V}^*(0) - \widehat{H}_1(1)\,S^*(0) &= v - \frac{\left(2-\sqrt{3}\right)}{\sqrt{3}} v \times 5, \\ &= \frac{6\sqrt{3}-10}{\sqrt{3}} v \end{aligned}$$

and

$$\begin{aligned} \widehat{H}_0(2,\omega) &= \widehat{V}^*(1,\omega) - \widehat{H}_1(2,\omega)\,S^*(1,\omega) \\ &= \begin{cases} 4\left(3+\sqrt{3}\right)^{-1} v - \left(6\sqrt{2}-8\right)\left(3+\sqrt{3}\right)^{-1} v \times 4 & \text{if } \omega \in A_{1,1} \\ 2\left(\sqrt{3}-1\right) v - \left(4-3\sqrt{2}\right)\left(\sqrt{3}-1\right) v \times 8 & \text{if } \omega \in A_{1,2} \end{cases} \\ &= \begin{cases} \left(36-24\sqrt{2}\right)\left(3+\sqrt{3}\right)^{-1} v & \text{if } \omega \in A_{1,1} \\ 6\left(\sqrt{3}-1\right)\left(4\sqrt{2}-5\right) v & \text{if } \omega \in A_{1,2} \end{cases} \end{aligned}$$