UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — An Introduction to Mathematical Finance

Day of examination: Monday 2. December 2019

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

The return of a zero coupon bond over 6 months is 3%. Find the implied continuous compounding annual rate.

b (weight 10p)

Let $C^E(0)$ and $C^A(0)$ be the prices at time zero of a European and an American call options, respectively, on a stock paying no dividends. Assume that both options have the same strike K and time to expiry T. Since the American option provides more optionality than the European one, it is clear that $C^A(0) \geq C^E(0)$ (you do not need to prove this inequality). Prove that, in fact, $C^E(0) = C^A(0)$.

c (weight 10p)

Explain how to construct a strangle. Write a table showing the profits given by this strategy in terms of the price of the stock at expiry time. You may assume that the interest rate is zero.

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Problem 2

Consider a one-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and prices given by $B(0) = 1, S_1(0) = 7,\ S_2(0) = 5$ and

$$B(1) = \begin{pmatrix} \frac{7}{6} \\ \frac{7}{6} \\ \frac{7}{6} \\ \frac{7}{6} \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} \frac{35}{3} \\ 7 \\ \frac{35}{6} \\ \frac{35}{3} \end{pmatrix} \quad S_2(1) = \begin{pmatrix} 7 \\ 7 \\ \frac{14}{3} \\ \frac{14}{3} \end{pmatrix}.$$

a (weight 10p)

Find all risk neutral measures in this market. Is this market arbitrage-free? Justify your answer.

b (weight 10p)

Find all contingent claims $X = (X_1, X_2, X_3, X_4)^T$ that are attainable in this market. Is this market complete? Justify your answer.

c (weight 10p)

Compute the arbritrage-free price (or prices) for the look-back option $X = \max(0, S_1(0) - 8, S_1(1) - 8)$.

Problem 3

Consider a two-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, probability measure $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$, interest rate $r = 0$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1,2}$ with prices given by

$$S_1(0) = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 2 \end{pmatrix}, \quad S_1(2) = \begin{pmatrix} 6 \\ 1 \\ 4 \\ 1 \end{pmatrix}.$$

a (weight 10p)

Find the filtration generated by the price process $S_1 = \{S_1(t)\}_{t=0,1,2}$. Discuss carefully the partitions associated to the price process and how they generate the algebras in the filtration.

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b (weight 20p)

Let $Q = \left(\frac{3}{10}, \frac{1}{5}, \frac{1}{6}, \frac{1}{3}\right)^T$ be the unique martingale measure on this market (you do not have to prove this). Consider the following optimal portfolio problem

$$\max_{H \in \mathbb{H}} \mathbb{E}\left[U\left(V\left(2\right)\right)\right]$$
$$\text{subject to } V\left(0\right) = v,$$

where v is a given non-negative real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U\left(u\right) = \log\left(u\right)$. Compute the optimal attainable wealth, the optimal expected utility and the optimal trading strategy.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Given $\mathcal{G}$ an algebra and X a random variable on Ω, give the abstract definition of conditional expectation of X given $\mathcal{G}$.

Define what is a martingale with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\ldots,T}$ under the probability measure P.

b (weight 10p)

Prove that if $\mathcal{H} \subset \mathcal{G}$ is also an algebra on Ω, then

$$\mathbb{E}\left[\mathbb{E}\left[X|\,\mathcal{G}\right]|\,\mathcal{H}\right] = \mathbb{E}\left[X|\,\mathcal{H}\right].$$

c (weight 10p)

Let Q be another probability measure on Ω that is equivalent to P, i.e., such that $Q > 0$. Define the process

$$L = \left\{L\left(t\right) = \mathbb{E}\left[\frac{Q}{P}|\mathcal{F}_t\right]\right\}_{t=0,\ldots,T}.$$

Let $X = \{X\left(t\right)\}_{t=0,\ldots,T}$ be a stochastic process. Show that X is a $\mathbb{F}$-martingale under Q if and only if $Z = \{Z\left(t\right) = L\left(t\right)X\left(t\right)\}_{t=0,\ldots,T}$ is a $\mathbb{F}$-martingale under P.

Hint: You may use without having to prove it that L is strictly positive and that if W is a random variable then

$$\mathbb{E}_Q\left[W|\mathcal{F}_t\right] = \frac{\mathbb{E}\left[WL\left(T\right)|\mathcal{F}_t\right]}{L\left(t\right)}.$$