UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — An Introduction to Mathematical Finance

Day of examination: Thursday 26. November 2020

Examination hours: 15.00 – 19.00

This problem set consists of 4 pages.

Appendices: All

Permitted aids: All

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

After how many days will a zero-coupon bond purchased at time zero for $B(0,1) = 0.93$ produce a 4% return? Assume there are 365 days in a year.

b (weight 10p)

Let $F(t,T), t \in [0,T]$ denote the forward price in a forward contract (on a non-paying dividends stock) starting at time t and with delivery date T. Let $V(t)$ be the time t value of a forward contract initiated at time zero. Show that if

$$V(t) < (F(t,T) - F(0,T))\, e^{-r(T-t)},$$

then you can make a risk free profit.

c (weight 10p)

Explain how to construct a butterfly spread. Write a table showing the profits given by this strategy in terms of the price of the stock at expiry time. You may assume that the interest rate is zero.

(Continued on page 2.)

Problem 2

Consider a one-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and prices given by $B(0) = 1, S_1(0) = 7,\ S_2(0) = 8$ and

$$B(1) = \begin{pmatrix} \frac{9}{8} \\ \frac{9}{8} \\ \frac{9}{8} \\ \frac{9}{8} \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 9 \\ \frac{45}{4} \\ \frac{27}{4} \\ \frac{27}{4} \end{pmatrix} \quad S_2(1) = \begin{pmatrix} \frac{27}{2} \\ \frac{27}{4} \\ \frac{27}{4} \\ \frac{45}{4} \end{pmatrix}.$$

a (weight 10p)

Find all risk neutral measures in this market. Is this market arbitrage-free? Justify your answer.

b (weight 10p)

Find all contingent claims $X = (X_1, X_2, X_3, X_4)^T$ that are attainable in this market. Is this market complete? Justify your answer.

c (weight 10p)

Compute the arbritrage-free price (or prices) for $X = \max(0, S_2(1) - S_1(1) - 9/4)$.

d (weight 5p)

Are the contingent claims of the form $Y = (Y_1, Y_2, Y_3, Y_4)^T$, for non-negative numbers $Y_1 = Y_2 = Y_3 \neq Y_4$, measurable with respecto to $\mathfrak{a}(S_1(1))$? And with respect to $\mathfrak{a}(S_2(1))$?

Problem 3

Consider a two-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, probability measure $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$, interest rate $r = 0$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1,2}$ with prices given by

$$S_1(0) = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 2 \end{pmatrix}, \quad S_1(2) = \begin{pmatrix} 6 \\ 1 \\ 4 \\ 1 \end{pmatrix}.$$

(Continued on page 3.)

a (weight 5p)

Find the filtration generated by the price process $S_1 = \{S_1(t)\}_{t=0,1,2}$. Discuss carefully the partitions associated to the price process and how they generate the algebras in the filtration.

b (weight 20p)

Let $Q = \left(\frac{3}{10}, \frac{1}{5}, \frac{1}{6}, \frac{1}{3}\right)^T$ be the unique martingale measure on this market (you do not have to prove this). Consider the following optimal portfolio problem

$$\max_{H \in \mathbb{H}} \mathbb{E}\left[U\left(V\left(2\right)\right)\right]$$
$$\text{subject to } V\left(0\right) = v,$$

where v is a given non-negative real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U(u) = 2u^{1/2}$. Compute the optimal attainable wealth, the optimal expected utility and the optimal trading strategy.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Let $\mathcal{G}$ be an algebra and X be a random variable on Ω.

Give the abstract definition of conditional expectation of X given $\mathcal{G}$.

Prove that the following two statements are equivalent:

1. The random variable Z is the conditional expectation of X given $\mathcal{G}$.
2. The random variable Z is $\mathcal{G}$-measurable and satisfies $\mathbb{E}\left[(X - Z)Y\right] = 0$, for all random variables Y that are $\mathcal{G}$-measurable.

b (weight 10p)

Define what is a martingale with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\ldots,T}$ under the probability measure P.

Let $\mathbb{G} = \{\mathcal{G}_t\}_{t=0,\ldots,T}$ be another filtration such that $\mathcal{G}_t \subseteq \mathcal{F}_t$. Prove that if M is an $\mathbb{F}$-martingale and M is $\mathbb{G}$-adapted, then M is also a $\mathbb{G}$-martingale.

c (weight 10p)

Consider the market model of Problem 3. Compute the variables of the stochastic process $A = \{A(t)\}_{t=0,1,2}$, such that

- $A(0) = 0$;

(Continued on page 4.)

- A is predictable with respect to $\mathbb{F}$, the filtration generated by the price process;
- $M = \left\{M\left(t\right) := S_1^2\left(t\right) - A\left(t\right)\right\}_{t=0,1,2}$ is a martingale with respect to $\mathbb{F}$.