UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — An Introduction to Mathematical Finance

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This problem set consists of 12 pages.

Appendices: All

Permitted aids: All

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

The price of a zero-coupon bond at time t is given by $B(t,T) = e^{-r(T-t)}$, where r is the implied annual (continuous) compounding rate. Moreover, the return of this bond bond over a period $[s,t] \subset [0,T]$ is given by

$$R(s,t) = \frac{B(t,T) - B(s,T)}{B(s,T)} = \frac{e^{-r(T-t)} - e^{-r(T-s)}}{e^{-r(T-s)}}.$$

Here, we have $s = 0, T = 1, B(0,1) = 0.93$ and $R(0,t) = 0.04$. Hence,

$$0.93 = B(0,1) = e^{-r(1-0)} \iff r = -\log(0.93) \simeq 0.0726 = 7.26\%.$$

On the other hand,

$$R(0,t) = \frac{e^{-r(1-t)} - B(0,1)}{B(0,1)},$$

which yields

$$\begin{aligned} t &= 1 + \frac{\log(B(0,1)(1 + R(0,t)))}{r} \\ &= 1 + \frac{\log(0.93(1 + 0.04))}{0.0726} \simeq 0.5406 \simeq 197.3190 \simeq 198 \text{ days}. \end{aligned}$$

(Continued on page 2.)

b (weight 10p)

Suppose that

$$V(t) < (F(t,T) - F(0,T))\, e^{-r(T-t)}. \tag{1}$$

Then, at time t:

- Borrow the amount $V(t)$.
- Pay $V(t)$ to enter a long forward position with forward price $F(0,T)$.
- Take a short forward position with forward price $F(t,T)$ (at no cost).

Next, at time T:

- Close the forward positions, getting:
- $S(T) - F(0,T)$ for the long position,
- $F(t,T) - S(T)$ for the short position.
- Pay $V(t)\, e^{r(T-t)}$ to settle the loan.

This will yield a risk free profit of

$$\begin{aligned} &S(T) - F(0,T) + F(t,T) - S(T) - V(t)\, e^{r(T-t)} \\ &= F(t,T) - F(0,T) - V(t)\, e^{r(T-t)} > 0. \end{aligned}$$

c (weight 10p)

Let $0 < K_1 < K_2 < K_3$. In this strategy you buy a call option with strike K_1 (for $C^E(0,K_1)$) and a call option with strike K_3 (for $C^E(0,K_3)$) and sell two call options with strike K_2 (for $2C^E(0,K_2)$). The profit of the strangle as a function of the final price of the stock $S(T)$ is given by

$$P(S(T)) = (S(T) - K_1)^+ + (S(T) - K_3)^+ - 2(S(T) - K_2)^+ - C,$$

where $C = C^E(0,K_1) + C^E(0,K_3) - 2C^E(0,K_2)$ is the initial cost of the strategy. In this case, the table of profits is given by

$S(T)$	Profit
$S(T) < K_1$	$-C$
$K_1 \le S(T) < K_2$	$S(T) - K_1 - C$
$K_2 \le S(T) < K_3$	$2K_2 - K_1 - S(T) - C$
$K_3 \le S(T)$	$2K_2 - K_1 - K_3 - C$

(Continued on page 3.)

Problem 2

a (weight 10p)

Let B denote the price process for the bank account. We have that $B(0) = 1$ and $\mathrm{B}(1) = \frac{9}{8}$. The discounted price processes for the risky assets are given by $S_1^*(0) = S_1(0)/B(0) = 7, S_2^*(0) = S_2(0)/B(0) = 8$, $S_1^*(1) = S_1(1)/B(1) = (8, 10, 6, 6)^T$ and $S_2^*(1)/B(1) = (12, 6, 6, 10)^T$. A risk neutral probability measure $Q{=}(Q_1, Q_2, Q_3, Q_4)^T$ must satisfy the following conditions

$$\mathbb{E}_Q[S_1^*(1)] = S_1^*(0),$$
$$\mathbb{E}_Q[S_2^*(1)] = S_2^*(0),$$

which are equivalent to the following equations

$$8Q_1 + 10Q_2 + 6Q_3 + 6Q_4 = 7, \tag{2}$$
$$6Q_1 + 3Q_2 + 3Q_3 + 5Q_4 = 4, \tag{3}$$
$$Q_1 + Q_2 + Q_3 + Q_4 = 1 \tag{4}$$

with the following restrictions $Q_1 > 0, Q_2 > 0, Q_3 > 0, Q_4 > 0$. From (4) we have that $Q_4 = 1 - Q_1 - Q_2 - Q_3$ and substituting this value in (2) and (3) we obtain

$$2Q_1 + 4Q_2 = 1, \tag{5}$$
$$Q_1 - 2Q_2 - 2Q_3 = -1. \tag{6}$$

From (5) we get that $Q_2 = \frac{1-2Q_1}{4}$. Substituting this value in (6) we get

$$Q_1 - 2\left(\frac{1-2Q_1}{4}\right) - 2Q_3 = -1 \Longleftrightarrow Q_3 = \frac{1+4Q_1}{4},$$

and

$$Q_4 = 1 - Q_1 - \frac{1-2Q_1}{4} - \frac{1+4Q_1}{4} = \frac{1-3Q_1}{2}.$$

Hence, setting $Q_1 = \lambda$, we get $Q_\lambda = \left(\lambda, \frac{1-2\lambda}{4}, \frac{1+4\lambda}{4}, \frac{1-3\lambda}{2}\right)^T$. Finally, using the restrictions $Q_i > 0, i = 1, ..., 4$, we have the following conditions on the parameter λ

$$Q_1 = \lambda > 0$$
$$Q_2 = \frac{1-2\lambda}{4} > 0 \Longleftrightarrow \lambda < \frac{1}{2},$$
$$Q_3 = \frac{1+4\lambda}{4} > 0 \Longleftrightarrow \lambda > -\frac{1}{4},$$
$$Q_4 = \frac{1-3\lambda}{2} > 0 \Longleftrightarrow \lambda < \frac{1}{3},$$

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which yield that $\lambda \in \left(0, \frac{1}{3}\right)$. Therefore, the set of risk neutral measures $\mathbb{M}$ is given by

$$\mathbb{M} = \left\{ Q_\lambda = \left(\lambda, \frac{1-2\lambda}{4}, \frac{1+4\lambda}{4}, \frac{1-3\lambda}{2} \right)^T, 0 < \lambda < \frac{1}{3} \right\}$$

By the first fundamental theorem of asset pricing we know that the market is arbitrage free because the set of risk neutral probability measures is non empty. Alternative parametrizations of $\mathbb{M}$ are

$$\begin{aligned} \mathbb{M} &= \left\{ Q_\lambda = \left(\frac{1-4\lambda}{2}, \lambda, \frac{3-8\lambda}{4}, \frac{12\lambda-1}{4} \right)^T, \frac{1}{12} < \lambda < \frac{3}{8} \right\} \\ &= \left\{ Q_\lambda = \left(\frac{4\lambda-1}{4}, \frac{3-4\lambda}{8}, \lambda, \frac{7-12\lambda}{8} \right)^T, \frac{1}{4} < \lambda < \frac{7}{12} \right\} \\ &= \left\{ Q_\lambda = \left(\frac{1-2\lambda}{3}, \frac{1+4\lambda}{12}, \frac{7-8\lambda}{12}, \lambda \right)^T, 0 < \lambda < \frac{1}{2} \right\}. \end{aligned}$$

b (weight 10p)

By the second fundamental theorem of asset pricing we can conclude that the market is not complete because there are infinitely many risk neutral measures in this market. A contingent claim $X = (X_1, X_2, X_3, X_4)^T$ is attainable if there exists a portfolio $H = (H_0, H_1, H_2)^T$ such that $X = H_0 B(1) + H_1 S_1(1) + H_2 S_2(1)$. This translates to the following system of equations

$$X_1 = \frac{9}{8} H_0 + 9 H_1 + \frac{27}{2} H_2, \tag{7}$$

$$X_2 = \frac{9}{8} H_0 + \frac{45}{4} H_1 + \frac{27}{4} H_2, \tag{8}$$

$$X_3 = \frac{9}{8} H_0 + \frac{27}{4} H_1 + \frac{27}{4} H_2, \tag{9}$$

$$X_4 = \frac{9}{8} H_0 + \frac{27}{4} H_1 + \frac{45}{4} H_2. \tag{10}$$

From (7) we get that $\frac{9}{8} H_0 = X_1 - 9H_1 - \frac{27}{2} H_2$. Substituting this expression for $\frac{9}{8} H_0$ in (8),(9) and (10) we obtain

$$X_2 = X_1 + \frac{9}{4} H_1 - \frac{27}{4} H_2, \tag{11}$$

$$X_3 = X_1 - \frac{9}{4} H_1 - \frac{27}{4} H_2, \tag{12}$$

$$X_4 = X_1 - \frac{9}{4} H_1 - \frac{9}{4} H_2 \tag{13}$$

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Solving (11) and (12) in H_1 and H_2 we get that

$$H_1 = \frac{2}{9}\left(X_2 - X_3\right),$$
$$H_2 = \frac{4X_1 - 2X_2 - 2X_3}{27},$$

and substituting these values in (13) we get

$$X_4 = X_1 - \frac{9}{4}\frac{2}{9}\left(X_2 - X_3\right) - \frac{9}{4}\frac{4X_1 - 2X_2 - 2X_3}{27} \Longleftrightarrow 2X_1 - X_2 + 2X_3 - 3X_4 = 0.$$

Alternatively, since $\mathbb{M} \neq \emptyset$, we have that X is attainable if and only if $\mathbb{E}_{Q_\lambda}\left[X/B\left(1\right)\right]$ does not depend on λ. We have that

$$\mathbb{E}_{Q_\lambda}\left[X/B\left(1\right)\right] = \frac{1}{B\left(1\right)}\left\{\lambda\left(X_1 - \frac{X_2}{2} + X_3 - \frac{3}{2}X_4\right) + \frac{X_2 + X_3 + 2X_4}{2}\right\},$$

and the previous expectation does not depend on λ if and only if

$$X_1 - \frac{X_2}{2} + X_3 - \frac{3}{2}X_4 = 0 \Longleftrightarrow 2X_1 - X_2 + 2X_3 - 3X_4 = 0.$$

c (weight 10p)

We have that

$$X = \begin{pmatrix} \max\left(0, S_2\left(1,\omega_1\right) - S_1\left(1,\omega_1\right) - 9/4\right) \\ \max\left(0, S_2\left(1,\omega_2\right) - S_1\left(1,\omega_2\right) - 9/4\right) \\ \max\left(0, S_2\left(1,\omega_3\right) - S_1\left(1,\omega_3\right) - 9/4\right) \\ \max\left(0, S_2\left(1,\omega_4\right) - S_1\left(1,\omega_4\right) - 9/4\right) \end{pmatrix} = \begin{pmatrix} \max\left(0, \frac{27}{2} - 9 - \frac{9}{4}\right) \\ \max\left(0, \frac{27}{4} - \frac{45}{4} - \frac{9}{4}\right) \\ \max\left(0, \frac{27}{4} - \frac{27}{4} - \frac{9}{4}\right) \\ \max\left(0, \frac{45}{4} - \frac{27}{4} - \frac{9}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{9}{4} \\ 0 \\ 0 \\ \frac{9}{4} \end{pmatrix},$$

and, therefore, it is not attainable because

$$2X_1 - X_2 + 2X_3 - 3X_4 = 2 \times \frac{9}{4} - 1 \times 0 + 2 \times 0 - 3 \times \frac{9}{4} = -\frac{9}{4} \neq 0.$$

Hence, there is an interval of arbitrage free prices $\left[V_-\left(X\right), V_+\left(X\right)\right]$, where $V_-\left(X\right)$ is the lower hedging price of X and $V_+\left(X\right)$ is the upper hedging price of X. Moreover, we know that

$$V_-\left(X\right) = \inf_{Q\in M}\left\{\mathbb{E}_Q\left[\frac{X}{B\left(1\right)}\right]\right\} = \inf_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B\left(1\right)}\right]\right\},$$

and

$$V_+\left(X\right) = \sup_{Q\in M}\left\{\mathbb{E}_Q\left[\frac{X}{B\left(1\right)}\right]\right\} = \sup_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B\left(1\right)}\right]\right\}.$$

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We have that

$$\begin{aligned}\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] &= \frac{8}{9}\mathbb{E}_{Q_\lambda}[X] \\ &= \frac{8}{9}\left\{\frac{9}{4}\times\lambda + 0\times\frac{1-2\lambda}{4} + 0\times\frac{1+4\lambda}{4} + \frac{9}{4}\times\frac{1-3\lambda}{2}\right\} \\ &= 2\left\{\lambda + \frac{1-3\lambda}{2}\right\} = 1-\lambda.\end{aligned}$$

The previous computation yields

$$\begin{aligned}V_-(X) &= \inf_{\lambda\in\left(0,\frac{1}{3}\right)}\{1-\lambda\} = \frac{2}{3}, \\ V_+(X) &= \sup_{\lambda\in\left(0,\frac{1}{3}\right)}\{1-\lambda\} = 1.\end{aligned}$$

d (weight 5p)

The algebra of events generated by $S_1(1)$, denoted by $\mathfrak{a}(S_1(1))$, is the algebra generated by the partition

$$\begin{aligned}\pi_1 &= \left\{\{S_1(1)=9\}, \left\{S_1(1)=\frac{45}{4}\right\}, \left\{S_1(1)=\frac{27}{4}\right\}\right\} \\ &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3,\omega_4\}\}.\end{aligned}$$

Moreover, a random variable (contingent claim) is measurable with respect to $\mathfrak{a}(S_1(1))$ if it is constant over the elements of π_1. In this case, since $Y_3 = Y(\omega_3) \neq Y(\omega_4) = Y_4$, Y is not constant over the elments of $\pi_1$and, hence, it is not measurable with respect to $\mathfrak{a}(S_1(1))$.

However, the algebra of events generated by $S_2(1)$, denoted by $\mathfrak{a}(S_2(1))$, is the algebra generated by the partition

$$\begin{aligned}\pi_2 &= \left\{\left\{S_2(1)=\frac{27}{2}\right\}, \left\{S_2(1)=\frac{27}{4}\right\}, \left\{S_1(1)=\frac{45}{4}\right\}\right\} \\ &= \{\{\omega_1\}, \{\omega_2,\omega_3\}, \{\omega_4\}\},\end{aligned}$$

and Y is constant over the elements of π_2.

Problem 3 (weight 5p)

a (weight 5p)

We first compute the partitions associated to $S_1(0), S_1(1)$ and $S_1(2)$. We have

$$\begin{aligned}&\pi_{S_1(0)} = \{S_1(0)=3\} = \{\Omega\}, \\ &\pi_{S_1(1)} = \{\{S_1(1)=2\}, \{S_1(1)=4\}\} = \{\{\omega_3,\omega_4\}, \{\omega_1,\omega_2\}\} =: \{A_{1,1}, A_{1,2}\}, \\ &\pi_{S_1(2)} = \{\{S_1(2)=1\}, \{S_1(2)=4\}, \{S_1(2)=6\}\} = \{\{\omega_2,\omega_4\}, \{\omega_3\}, \{\omega_1\}\} =: \{A_{2,1}, A_{2,2}, A_{2,3}\}.\end{aligned}$$

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The partitions associated to $(S_1(0), S_1(1))$ and to $(S_1(0), S_1(1), S_1(2))$ are given by

$$\begin{aligned}
\pi_{(S_1(0),S_1(1))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} = \{\Omega \cap A_{1,1}, \Omega \cap A_{1,1}\} = \{A_{1,1}, A_{1,2}\}, \\
\pi_{(S_1(0),S_1(1),S_1(2))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} \cap \pi_{S_1(2)} = \pi_{S_1(0),S_1(1)} \cap \pi_{S_1(2)} \\
&= \{A_{1,1} \cap A_{2,1}, A_{1,1} \cap A_{2,2}, A_{1,1} \cap A_{2,3}, A_{1,2} \cap A_{2,1}, A_{1,2} \cap A_{2,2}, A_{1,2} \cap A_{2,3}\} \\
&= \{\{\omega_4\}, \{\omega_3\}, \emptyset, \{\omega_2\}, \emptyset, \{\omega_1\}\} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}.
\end{aligned}$$

The filtrations are given by

$$\begin{aligned}
&\mathcal{F}_0 = \mathfrak{a}(S_1(0)) = \mathfrak{a}(\{\Omega\}) = \{\emptyset, \Omega\}, \\
&\mathcal{F}_1 = \mathfrak{a}(S_1(0), S_1(1)) = \mathfrak{a}(\{A_{1,1}, A_{1,2}\}) = \{\emptyset, \Omega, A_{1,1}, A_{1,2}\} = \{\emptyset, \Omega, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2\}\}, \\
&\mathcal{F}_2 = \mathfrak{a}(S_1(0), S_1(1), S_1(2)) = \mathfrak{a}(\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}) = \mathcal{P}(\Omega),
\end{aligned}$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω.

b (weight 20p)

Since $\mathbb{M} = \{Q\}$ the market is arbitrage free and complete, due to the first and second fundamental theorem of asset pricing. Then, we can use the martingale method to solve the optimal portfolio problem. In this setup, $\mathbb{M} = \{Q\}$, the martingale method consists in the following two steps:

1. We first solve the constrained optimization problem

$$\max_W \mathbb{E}[U(W)]$$
$$\text{subject to } \mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v,$$

   and obtain the optimal attainable wealth $\widehat{W}$.

2. Given $\widehat{W}$, we find the optimal trading strategy $\widehat{H}$ such that its associated value process $\widehat{V}$ replicates $\widehat{W}$, that is, $\widehat{V}(2) = \widehat{W}$.

The previous constrained problem can be solved using the Lagrange multipliers method. The optimal attainable wealth $\widehat{W}$ is given by

$$\widehat{W} = I\left(\frac{\widehat{\lambda} L}{B(2)}\right),$$

where I is the inverse of $U'(u)$, L is the state-price density vector $L = \frac{Q}{P}$, $B(2)$ is the price of the risk-less asset at time 2 and $\widehat{\lambda}$ is the optimal Lagrange multiplier associated to the constraint $\mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v$. Taking into account that $r = 0$,$U(u) = 2u^{1/2}$,

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$P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$ and $Q = \left(\frac{3}{10}, \frac{1}{5}, \frac{1}{6}, \frac{1}{3}\right)^T$,we have that

$$\begin{aligned} i = U'(u) = u^{-1/2} &\Longleftrightarrow I(i) = u = i^{-2}, \\ L = \left(\frac{\frac{3}{10}}{\frac{1}{4}}, \frac{\frac{1}{5}}{\frac{1}{4}}, \frac{\frac{1}{6}}{\frac{1}{4}}, \frac{\frac{1}{3}}{\frac{1}{4}}\right)^T &= \left(\frac{6}{5}, \frac{4}{5}, \frac{2}{3}, \frac{4}{3}\right)^T, \\ B(2) = 1, \end{aligned}$$

which yield $\widehat{W} = \left(\widehat{\lambda} L\right)^{-2}$. The optimal Lagrange multiplier $\widehat{\lambda}$ satisfies the equation

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B(2)}\right] = \mathbb{E}_Q\left[\frac{I\left(\frac{\widehat{\lambda} L}{B(2)}\right)}{B(2)}\right] = \mathbb{E}_Q\left[\left(\widehat{\lambda} L\right)^{-2}\right] = \left(\widehat{\lambda}\right)^{-2} \mathbb{E}_Q\left[L^{-2}\right].$$

Therefore, we get

$$\widehat{\lambda} = \left(\frac{\mathbb{E}_Q\left[L^{-2}\right]}{v}\right)^{1/2}, \qquad \widehat{W} = v\frac{L^{-2}}{\mathbb{E}_Q\left[L^{-2}\right]},$$

and the optimal objective value is given by

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right] = \mathbb{E}\left[2\left(v\frac{L^{-2}}{\mathbb{E}_Q\left[L^{-2}\right]}\right)^{1/2}\right] = 2v^{1/2}\frac{\mathbb{E}\left[L^{-1}\right]}{\mathbb{E}_Q\left[L^{-2}\right]^{1/2}} = 2v^{1/2}\mathbb{E}_Q\left[L^{-2}\right]^{1/2},$$

where we have used that $\mathbb{E}\left[L^{-1}\right] = \mathbb{E}\left[LL^{-2}\right] = \mathbb{E}_Q\left[L^{-2}\right]$. Moreover,

$$\begin{aligned} \mathbb{E}_Q\left[L^{-2}\right] = \mathbb{E}\left[L^{-1}\right] &= \frac{1}{4}\left\{\left(\frac{6}{5}\right)^{-1} + \left(\frac{4}{5}\right)^{-1} + \left(\frac{2}{3}\right)^{-1} + \left(\frac{4}{3}\right)^{-1}\right\} \\ &= \frac{1}{4}\left\{\frac{5}{6} + \frac{5}{4} + \frac{3}{2} + \frac{3}{4}\right\} = \frac{13}{12}, \end{aligned}$$

and

$$L^{-2} = \left(\left(\frac{6}{5}\right)^{-2}, \left(\frac{4}{5}\right)^{-2}, \left(\frac{2}{3}\right)^{-2}, \left(\frac{4}{3}\right)^{-2}\right)^T = \left(\frac{25}{36}, \frac{25}{16}, \frac{9}{4}, \frac{9}{16}\right)^T.$$

Hence, we obtain $\mathbb{E}\left[U\left(\widehat{W}\right)\right] = 2v^{1/2}\left(\frac{13}{12}\right)^{1/2}$ and

$$\widehat{W} = \frac{v}{\frac{13}{12}}\left(\frac{25}{36}, \frac{25}{16}, \frac{9}{4}, \frac{9}{16}\right)^T = \left(\frac{25}{39}v, \frac{75}{52}v, \frac{27}{13}v, \frac{27}{52}v\right)^T.$$

Finally, we have to compute the optimal trading strategy $\widehat{H} = \left\{\left(\widehat{H}_0(t), \widehat{H}_1(t)\right)^T\right\}_{t=1,2}$, that is, a self-financing and predictable process such that its asociated value process $\widehat{V}$

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satisfies $\widehat{V}(2) = \widehat{W}$. We first compute, taking into account that $r = 0$, the discounted increments of the risky asset

$$\Delta S_1^*(2) = \Delta S_1(2) = (2, -3, 2, -1)^T,$$
$$\Delta S_1^*(1) = \Delta S_1(1) = (1, 1, -1, -1)^T.$$

- For $t = 2$, using that $\widehat{H}$ must be self-financing we have that $\widehat{W} = \frac{\widehat{W}}{B(2)} = \widehat{V}^*(1) + \widehat{H}_1(2)\,\Delta S_1^*(2)$.
  - Assuming that $\omega \in A_{1,1} = \{\omega_3, \omega_4\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned} \frac{27}{13}v = \widehat{W}_3 &= \widehat{V}^*(1,\omega_3) + \widehat{H}_1(2,\omega_3) \times 2, \\ \frac{27}{52}v = \widehat{W}_4 &= \widehat{V}^*(1,\omega_4) + \widehat{H}_1(2,\omega_4) \times (-1), \\ \widehat{V}^*(1,\omega_3) &= \widehat{V}^*(1,\omega_4), \\ \widehat{H}_1(2,\omega_3) &= \widehat{H}_1(2,\omega_4), \end{aligned}$$

    which, using that $r = 0$, yield

$$\begin{aligned} \widehat{V}^*(1,\omega_3) = \widehat{V}^*(1,\omega_4) = \widehat{V}(1,\omega_3) = \widehat{V}(1,\omega_4) &= \frac{27}{26}v, \\ \widehat{H}_1(2,\omega_3) = \widehat{H}_1(2,\omega_4) &= \frac{27}{52}v. \end{aligned}$$

  - Assuming that $\omega \in A_{1,2} = \{\omega_1, \omega_2\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned} \frac{25}{39}v = \widehat{W}_1 &= \widehat{V}^*(1,\omega_1) + \widehat{H}_1(2,\omega_1) \times 2, \\ \frac{75}{52}v = \widehat{W}_2 &= \widehat{V}^*(1,\omega_2) + \widehat{H}_1(2,\omega_2) \times (-3), \\ \widehat{V}^*(1,\omega_1) &= \widehat{V}^*(1,\omega_2), \\ \widehat{H}_1(2,\omega_1) &= \widehat{H}_1(2,\omega_2), \end{aligned}$$

    which, using that $r = 0$, yield

$$\begin{aligned} \widehat{V}^*(1,\omega_1) = \widehat{V}^*(1,\omega_2) = \widehat{V}(1,\omega_1) = \widehat{V}(1,\omega_2) &= \frac{25}{26}v, \\ \widehat{H}_1(2,\omega_1) = \widehat{H}_1(2,\omega_2) &= -\frac{25}{156}v. \end{aligned}$$

- For $t = 1$, the predictability assumption yields that $\widehat{H}_1(1)$ is constant. Moreover, using that $\widehat{H}$ must be self-financing we have that $\widehat{V}^*(1) = \widehat{V}^*(0) + \widehat{H}_1(1)\,\Delta S_1^*(1)$

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and we get the following two equations

$$\frac{27}{26}v = \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (-1), \qquad (\text{for } \omega \in A_{1,1})$$
$$\frac{25}{26}v = \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (1), \qquad (\text{for } \omega \in A_{1,2})$$

which, using that $r = 0$, yield

$$\widehat{V}^*(0) = \widehat{V}(0) = v, \qquad \widehat{H}_1(1) = -\frac{1}{26}v.$$

- Finally we compute $H_0(1)$ and $H_0(2)$ from the definition of value process. We have

$$\widehat{H}_0(1) = \widehat{V}^*(0) - \widehat{H}_1(1)\, S_1^*(0) = v + \frac{1}{26}v \times 3 = \frac{29}{26}v,$$

and

$$\widehat{H}_0(2,\omega) = \widehat{V}^*(1,\omega) - \widehat{H}_1(2,\omega)\, S_1^*(1,\omega)$$
$$= \begin{cases} \frac{27}{26}v - \frac{27}{52}v \times 2 = 0 & \text{if } \omega \in A_{1,1} \\ \frac{25}{26}v + \frac{25}{156}v \times 4 = \frac{125}{78}v & \text{if } \omega \in A_{1,2} \end{cases}$$

Problem 4

a (weight 10p)

The conditional expectation of X given $\mathcal{G}$ is the unique random variable $\mathbb{E}[X|\mathcal{G}]$ satisfying:

1. $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$- measurable.
2. $\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_B], \qquad B \in \mathcal{G}.$

$1 \Rightarrow 2$) That Z is $\mathcal{G}$-measurable follows from the $\mathcal{G}$-measurability of $\mathbb{E}[X|\mathcal{G}]$. Moreover, we can reason as follows

$$\mathbb{E}[(X-Z)Y] \overset{(a)}{=} \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])Y] \overset{(b)}{=} \mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])Y|\mathcal{G}]],$$
$$\overset{(c)}{=} \mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])|\mathcal{G}]Y] \overset{(d)}{=} \mathbb{E}[\mathbb{E}[(\mathbb{E}[X|\mathcal{G}] - \mathbb{E}[X|\mathcal{G}])|\mathcal{G}]Y] = 0,$$

where we have used that: (a)By assumption, (b)Law of total expectation, (c) what is $\mathcal{G}$-measurable goes out, (d)Linearity of conditional expectation and what is $\mathcal{G}$-measurable goes out again.

$2 \Rightarrow 1$) Property 1. of conditional expectation follows by assumption. Property 2. follows by taking $Y = \mathbf{1}_B, B \in \mathcal{G}$. Then,

$$0 = \mathbb{E}[(X-Z)Y] = \mathbb{E}[(X-Z)\mathbf{1}_B] \Longleftrightarrow \mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[Z\mathbf{1}_B].$$

Since B is an arbitrary set in $\mathcal{G}$, the previous equality shows that Z satisfies property 2. of the conditional expectation.

(Continued on page 11.)

b (weight 10p)

A process M is an $\mathbb{F}$-martingale if M is $\mathbb{F}$-adapted and satisfies

$$\mathbb{E}\left[M(t+1)|\mathcal{F}_t\right] = M(t), \qquad t = 0, ..., T-1.$$

Let M be a $\mathbb{G}$-adapted process that is an $\mathbb{F}$-martingale, with $\mathcal{G}_t \subseteq \mathcal{F}_t$. To prove that M is also a $\mathbb{G}$-martingale we only need to prove the martingale property because by assumption is $\mathbb{G}$-adapted. Then,

$$\mathbb{E}\left[M(t+1)|\mathcal{G}_t\right] \overset{(a)}{=} \mathbb{E}\left[\mathbb{E}\left[M(t+1)|\mathcal{F}_t\right]|\mathcal{G}_t\right] \overset{(b)}{=} \mathbb{E}\left[M(t)|\mathcal{G}_t\right] \overset{(c)}{=} M(t), \qquad t = 0, ..., T-1,$$

where we have used that: (a)The tower property of conditional expectation and $\mathcal{G}_t \subseteq \mathcal{F}_t$, $(b)M$ is an $\mathbb{F}$-martingale, (c) M is $\mathbb{G}$-adapted ($M(t)$ is $\mathcal{G}_t$-measurable) and the property that if Z is a $\mathcal{G}$-measurable random variable then $Z = \mathbb{E}\left[Z|\mathcal{G}\right]$.

c (weight 10p)

Note that

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \qquad \mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2\}\}, \qquad \mathcal{F}_2 = \mathcal{P}(\Omega).$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω. Therefore, the predictability constraint on the process A implies that $A(1) = a_{1,1}$ (a constant) and $A(2,\omega) = a_{2,1}\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + a_{2,2}\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega)$. The square of the price process is given by

$$\begin{aligned} S_1^2(0) &= 9 \\ S_1^2(1,\omega) &= 16\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + 4\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega), \\ S_1^2(2,\omega) &= 36\mathbf{1}_{\{\omega_1\}}(\omega) + 16\mathbf{1}_{\{\omega_3\}}(\omega) + 1\mathbf{1}_{\{\omega_2,\omega_4\}}(\omega). \end{aligned}$$

The process A is $\mathbb{F}$-adapted because it is $\mathbb{F}$-predictable. This yields that M_t is $\mathcal{F}_t$-measurable because it is a function of the two $\mathcal{F}_t$-measurable random variables S_t and A_t. Hence, M is $\mathbb{F}$-adapted. Now we only need to prove the martingale property, which boils down to check

$$M(0) = \mathbb{E}\left[M(1)|\mathcal{F}_0\right] = \mathbb{E}\left[M(1)\right] \tag{14}$$

and

$$M(1) = \mathbb{E}\left[M(2)|\mathcal{F}_1\right]. \tag{15}$$

Since $A(0) = 0$, we have that $M(0) = S_1^2(0) - A(0) = S_1^2(0) = 9$ and

$$\mathbb{E}\left[M(1)\right] = \mathbb{E}\left[S_1^2(1)\right] - A(1) = 16(P_1 + P_2) + 4(P_3 + P_4) - a_{1,1}.$$

Hence, using equation (14), we get

$$A(1) = a_{1,1} = 16(P_1 + P_2) + 4(P_3 + P_4) - 9.$$

(Continued on page 12.)

On the other hand,

$$\begin{aligned} M(1) &= S_1^2(1) - A(1) \\ &= (16 - a_{1,1})\,\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + (4 - a_{1,1})\,\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega), \\ M(2) &= S_1^2(2) - A(2) \\ &= (36 - a_{2,1})\,\mathbf{1}_{\{\omega_1\}}(\omega) + (1 - a_{2,1})\,\mathbf{1}_{\{\omega_2\}}(\omega) \\ &\qquad + (16 - a_{2,2})\,\mathbf{1}_{\{\omega_3\}}(\omega) + (1 - a_{2,2})\,\mathbf{1}_{\{\omega_4\}}(\omega) \end{aligned}$$

and

$$\mathbb{E}[M(2)|\,\mathcal{F}_1] = \mathbb{E}[M(2)|\,\{\omega_1,\omega_2\}]\,\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + \mathbb{E}[M(2)|\,\{\omega_3,\omega_4\}]\,\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega).$$

Moreover,

$$\begin{aligned} \mathbb{E}[M(2)|\,\{\omega_1,\omega_2\}] = \mathbb{E}[M(2)|\,\{\omega_1,\omega_2\}] &= \frac{(36 - a_{2,1})\,P_1 + (1 - a_{2,1})\,P_2}{P_1 + P_2} \\ &= \frac{36P_1 + P_2}{P_1 + P_2} - a_{2,1}, \\ \mathbb{E}[M(2)|\,\{\omega_3,\omega_4\}] = \mathbb{E}[M(2)|\,\{\omega_3,\omega_4\}] &= \frac{(16 - a_{2,2})\,P_3 + (1 - a_{2,2})\,P_4}{P_3 + P_4} \\ &= \frac{16P_3 + P_4}{P_3 + P_4} - a_{2,2}, \end{aligned}$$

and, therefore,

$$\mathbb{E}[M(2)|\,\mathcal{F}_1] = \left(\frac{36P_1 + P_2}{P_1 + P_2} - a_{2,1}\right)\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + \left(\frac{16P_3 + P_4}{P_3 + P_4} - a_{2,2}\right)\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega).$$

Finally, using equation (15), we get

$$\begin{aligned} a_{2,1} &= \frac{36P_1 + P_2}{P_1 + P_2} - 16 + a_{1,1}, \\ a_{2,2} &= \frac{16P_3 + P_4}{P_3 + P_4} - 4 + a_{1,1}. \end{aligned}$$

If we take $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$ we get

$$\begin{aligned} a_{1,1} &= 16 \times \frac{1}{2} + 4 \times \frac{1}{2} - 9 = 1, \\ a_{2,1} &= \frac{36 \times \frac{1}{4} + \frac{1}{4}}{\frac{1}{2}} - 16 + 1 = \frac{37}{2} - 15 = \frac{7}{2}, \\ a_{2,2} &= \frac{16 \times \frac{1}{4} + \frac{1}{4}}{\frac{1}{2}} - 4 + 1 = \frac{17}{2} - 3 = \frac{11}{2}. \end{aligned}$$