Proof of lemma 5.11

1) $\exists$ a DTS
2) $\exists$ a trad. strategy satisfying $\begin{cases} V(0) = 0 \\ V(1,\omega) > 0, \ \omega \in \Omega \end{cases}$ (5.9)
3 $\exists$ a trad. strategy " $\begin{cases} V(0) < 0 \\ V(1,\omega) \geq 0, \ \omega \in \Omega \end{cases}$ (5.10)

We will show 1) $\iff$ 2) and 2) $\iff$ 3).

$\boxed{1) \iff 2)}$

1) $\Leftarrow$ 2) A strategy satisfying (5.9) is dominant. In particular, it dominates the strategy $\tilde{H} = (0, 0, \dots 0)^T$, which satisfies $\hat{V}(0) = 0$ and $\tilde{V}(1,\omega) = 0 < V(1,\omega), \omega \in \Omega$

1) $\Rightarrow$ 2) Suppose that the strategy $\hat{H}$ dominates the strategy $\tilde{H}$. Then, consider a new strategy $H := \hat{H} - \tilde{H}$. By linearity of V as a function of H one has that

$V(0) = \hat{V}(0) - \tilde{V}(0) = 0 - 0 = 0$ and

$V(1,\omega) = \hat{V}(1,\omega) - \tilde{V}(1,\omega) \underset{\text{as } \hat{H} \text{ dominates } \tilde{H}}{>} 0, \ \omega \in \Omega$

$\boxed{2) \Leftrightarrow 3)}$

$2) \Rightarrow 3)$) Suppose that the trad. strategy H is such that $V(0) = 0$ and $V(1,\omega) > 0$, $\omega \in \Omega$. Since $B(t,\omega) > 0$ $\forall \omega \in \Omega$ and $t = 0, 1$ one has that $V^*(0) = \frac{V(0)}{B(0)} = 0$ and $V^*(1,\omega) = \frac{V(1,\omega)}{B(1)} > 0$ $\forall \omega \in \Omega$.

Since $V^*(1) = V^*(0) + G^* \Rightarrow H = (H_0, H_1, \dots H_N)^T$ must satisfy $G^*(\omega) = \sum_{n=1}^{N} H_n \Delta S_n^*(\omega) > 0$.

So it makes sense to consider the quantity

$$\delta := \min_{\omega \in \Omega} G^*(\omega) > 0$$

Introduce the strategy $\hat{H} = (\hat{H}_0, \hat{H}_1, \dots \hat{H}_N)^T$ satisfying

$$\tilde{H}_n := H_n, \quad n = 1, \dots, N$$

$$\tilde{H}_0 = - \sum_{n=1}^{N} H_n S_n^*(0) - \delta$$

Then, using that $\hat{V}^*(t) = \hat{H}_0 + \sum_{n=1}^{N} \hat{H}_n S_n^*(t)$ we get

$$\tilde{V}^*(0) = \tilde{H}_0 + \sum_{n=1}^{N} H_n S^*(0) = -\delta < 0$$

and $\tilde{V}^*(1,\omega) = \tilde{V}^*(0) + \tilde{G}^*(\omega) = -\delta + \hat{G}^*(\omega) \geq 0 \quad \forall \omega \in \Omega.$

By the def of δ

3) $\Rightarrow$ 2)

Suppose that the trad. strat. $\hat{H}$ is such that $\hat{V}(0) < 0$ and $V(1,\omega) \geq 0$ for all $\omega \in \Omega$. Again, due to $B(1,\omega) > 0$, we have that

$$\hat{V}^*(0) = \frac{V(0)}{B(0)} < 0 \quad \text{and} \quad \hat{V}^*(1,\omega) = \frac{V(1,\omega)}{B(1)} \geq 0 \ \forall \omega \in \Omega.$$

Since $\hat{V}^*(1,\omega) = \hat{V}^*(0) + \hat{G}^*(\omega)$, $\omega \in \Omega$, $\hat{H}$ must satisfy that $\hat{G}^* > 0 \ \forall \omega \in \Omega$.

Introduce the strategy $H = (H_0, H_1, \dots, H_N)^T$ with

$$H_n = \hat{H}_n \quad n = 1, \dots, N.$$

$$H_0 = - \sum_{n=1}^{N} \hat{H}_n S_n^*(0)$$

Then,

$$V^*(0) = H_0 + \sum_{n=1}^{N} \hat{H}_n S_n^*(0) = 0$$

$$V^*(1,\omega) = V^*(0) + G^*(\omega) = 0 + \hat{G}^* > 0, \ \forall \omega \in \Omega.$$

Proof 5.13

1) Let π be a LPM.

Take $H = (1, 0, \dots, 0)$. Then, equation (5.12)

$$H_0 + \sum_{n=1}^{N} H_n S_n^*(0) = \sum_{\omega \in \Omega} \pi(\omega) \left(H_0 + \sum_{n=1}^{N} H_n S_n^*(1, \omega) \right)$$

$$H_0 = \sum_{\omega \in \Omega} \pi(\omega) H_0 \iff \sum_{\omega \in \Omega} \pi(\omega) = 1$$

Combined with $\pi \geq 0$, we can conclude.

2) $\Rightarrow$) By 1) π is a prob. measure. For $i = 1, \dots, N$

the strategy $H(i) = (\delta_{0i}, \delta_{1i}, \dots, \delta_{Ni})$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$. Then, using equation (5.12)

we get $S_i^*(0) = \sum_{\omega \in \Omega} \pi(\omega) S_i^*(\omega)$.

($\Leftarrow$) Suppose that π is a prob measure and (5.13) holds.
Then π is a non-negative and

$$
\begin{aligned}
V^*(0) &= H_0 + \sum_{n=1}^{N} H_n S^*_n(0) \\
&= H_0 \underbrace{\Big(\sum_{\omega\in\Omega} \pi(\omega) \Big)}_{=1} + \sum_{n=1}^{N} H_n \Big(\sum_{\omega} S^*_n(1,\omega)\, \pi(\omega) \Big) \\
&= \sum_{\omega\in\Omega} \pi(\omega) \Big(H_0 + \sum_{n=1}^{N} H_n S^*(1,\omega) \Big) \\
&= \sum_{\omega\in\Omega} \pi\, V^*(1,\omega)
\end{aligned}
$$

That is (5.11) is satisfied.

Proof of lemma 5.16

The proof will be based on linear programming and duality theory.

Let $\bar{S}^*(0) = (B^*(0), S_1^*(0), \dots, S_N^*(0))^T$ and consider the linear programming problem

$$(P) \quad \max\{ 0^T \pi : S^{*T}(1,\Omega)\pi = \bar{S}^*(0), \pi \geq 0 \}$$

with associated dual problem given by

$$(D) \quad \min\{ \bar{S}^{*T}(0) H : S^*(1,\Omega) H \geq 0 \}$$

$\Rightarrow$) Let $\hat{\pi} = (\hat{\pi}_1, \dots, \hat{\pi}_K)^T \in \mathbb{R}^K$ be a LPM.

Then, we have that $\hat{\pi} \geq 0$ and

$$\bar{S}^*(0) = E_{\hat{\pi}}[\bar{S}^*(1)] = S^{*T}(1,\Omega)\hat{\pi}$$

Hence $\hat{\pi}$ belongs to the feasible set for (P).

On the other hand the optimal value of (P) is 0 and it is obviously attained by $\hat{x}$.

By strong duality, we have that $\exists$ an optimal solution $\hat{H}$ for (D) such that has the same optimal value as the primal problem (P), that is,

$$0 = (\bar{S}^{*T}(0))\hat{H} = 1\hat{H}_0 + S_1^*(0)\hat{H}_1 + \dots + S_N^*(0)\hat{H}_N = \hat{V}^*(0)$$

$$0 \leq S^*(1,\Omega)\hat{H} = \hat{V}^*(1) \Rightarrow V^*(1,\omega) \geq 0 \quad \omega \in \Omega.$$

In addition, since the optimal value is 0, we can conclude that $\nexists$ H such that $V^*(1) \geq 0$ and $V^*(0) < 0$, because it will contradict that $\hat{H}$ is optimal for (D).

By Lemma 5.11, we can conclude $\nexists$ DTS.

($\Leftarrow$) Suppose $\nexists$ DTS. By Lemma 5.11, this is equivalent to $\nexists$ of a trading strategy H satisfying $V^*(0) < 0$ and $V^*(1) \geq 0$.
This means that 0 is a lower bound for the optimal value of (D). Indeed, the strategy $H = (0, \dots, 0)^T$ is a solution of (D).
By strong duality, again, we can conclude that $\exists\ \hat{\pi}$ solving the problem (P) with optimal value 0.
This means that $\hat{\pi}$ satisfies

$$\left(B^*(0), S_1^*(0), \dots, S_N^*(0)\right)^T = \bar{S}^*(0) = S^{*T}(1, \Omega)\,\hat{\pi}$$
$$= E_{\hat{\pi}}\left[\bar{S}^*(1)\right]$$

and $\hat{\pi} \geq 0 \Rightarrow \hat{\pi}$ is a LPM.