Proof of Lemma 26

H is an A.O $\iff$ a) $G^*(\omega) \geq 0$, $\omega \in \Omega$

b) $E[G^*] > 0$

H is an A.O. $\iff$ 1) $V^*(0) = 0$

2) $V^*(1,\omega) \geq 0$, $\omega \in \Omega$

3) $E[V^*(1)] > 0$

$\Rightarrow$) Suppose H is an A.O.. Then,

$$0 \underset{(2)}{\leq} V^*(1,\omega) = V^*(0) + G^*(\omega) \underset{1)}{=} G^*(\omega), \ \omega \in \Omega \quad \text{(a) ✓}$$

and

$$0 \underset{3)}{<} E[V^*(1)] = \underbrace{E[V^*(0)]}_{\overset{1)}{=} 0} + E[G^*] = E[G^*] \quad \text{(b) ✓}$$

$\Leftarrow$) Suppose a) and b) hold for some strategy $\hat{H}$.

Define the strategy $H = (H_0, H_1, \dots, H_N)^T$ by

$$H_n = \hat{H}_n, \ n = 1, \dots, N \quad \text{and} \quad H_0 = -\sum_{n=1}^{N} H_n S_n^*(0)$$

Then,

$$V^*(0) = H_0 + \sum_{n=1}^{N} H_n S_n^*(0) = 0 \quad \text{(1) ✓}$$

$$V^*(1,\omega) = \underbrace{V^*(0)}_{=0} + G^*(\omega) = G^*(\omega) = \hat{G}^*(\omega) \geq 0, \ \omega \in \Omega \quad \text{(2) ✓}$$

and

$$E[V^*(1)] = \underbrace{E[V^*(0)]}_{=0} + E[G^*] = E[\hat{G}^*] > 0 \quad \text{(3) ✓}$$

<u>Proof of Thm 36</u> (FFTAP)

$$\nexists \text{ A.O} \iff \exists \text{ RNPM}$$

Consider the set

$$W = \{ X \in \mathbb{R}^K : X = G^* \text{ for some trading strategy } H \}$$

<u>Obs</u>

W is a set of rand. variables.

W is the set of all possible discounted value processes of strategies with zero initial investment.

Moreover, W is linear subspace of $\mathbb{R}^K$.

Next, consider the set

$$A = \{ X \in \mathbb{R}^K : X \geq 0, X \neq 0 \} \quad \left(\text{Non-negative orthant of } \mathbb{R}^K \right)$$

By Lemma 26

(*) $\exists$ A.O. $\iff$ $W \cap A \neq \emptyset$ $\iff$ $\exists \omega \in \Omega : G^*(\omega) > 0$

$W \cap A = \{ X \in \mathbb{R}^K : X = G^* \geq 0, X \neq 0 \}$

Consider the orthogonal complement of W, that is,

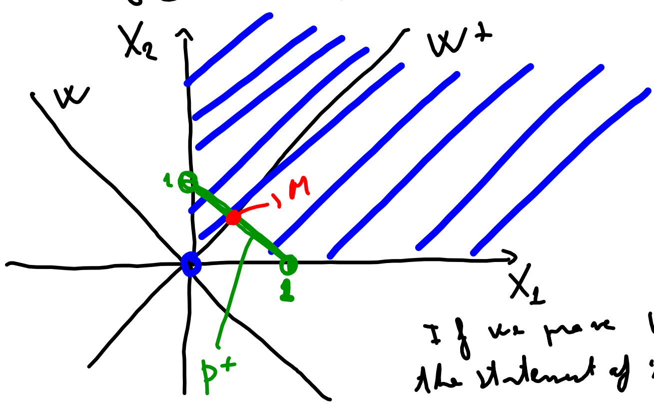
$$W^{\perp} = \{ y \in \mathbb{R}^K : x^T y = 0, \forall x \in W \} \quad \text{(subspace)}$$

Consider also the set of probability measures equivalent to P

$$P^{+} := \{ x \in \mathbb{R}^K : x_1 + x_2 + \dots + x_K = 1, \; x_1 > 0, \dots, x_K > 0 \}$$

and denote by $\mathcal{M}$ the set of all RNPM, which is a subset of P^+.

Geometric heuristics $\quad K = 2 \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



From the picture, it seems obvious that

$$(**) \quad W \cap A = \emptyset \iff W^{\perp} \cap P^{+} \neq \emptyset$$

$$\left(W \cap A \neq \emptyset \iff W^{\perp} \cap P^{+} = \emptyset \right)$$

If we prove $(**)$ and $M = W^{\perp} \cap P^{+}$, then the statement of the theorem holds thanks to $(*)$

Proof of $M = W^\perp \cap P^+$

$\subseteq$) Let $Q \in M$, then for any $G^* \in W$ we have that

$$E_Q[G^*] = E_Q\Big[\sum_{n=1}^{N} H_n \Delta S_n^*\Big] = \sum_{n=1}^{N} H_n E_Q[\Delta S_n^*] \underset{\substack{\Uparrow \\ Q \in M}}{=} 0$$

$\Rightarrow$ $Q \in W^\perp$

Since $Q_k > 0$, $k = 1, \dots K$ and $\sum_{k=1}^{K} Q_k = 1$, we can conclude that $Q \in W^\perp \cap P^+$

$\supseteq$) Let $Q \in W^\perp \cap P^+$. Then, Q is a probability measure equivalent to P. Hence, we only to check that

$$E_Q[\Delta S_n^*] = \sum_{k=1}^{K} \Delta S_n^* Q_k = 0 \qquad n = 1, \dots, N.$$

But this holds because $\Delta S_n^* = \sum_{m=1}^{N} \delta_{n,m} \Delta S_m^* \in W$ and $Q \in W^\perp$

Proof $\quad W \cap A = \emptyset \iff \emptyset \neq W^{\perp} \cap P^{+} = M$

$\Leftarrow$) Let $Q \in M$. Suppose that $G^* \in W \cap A$.

that is, $G^*(\omega_k) \geq 0$ $k = 1, \dots, K$ and $\exists$ at least one k_0 such that $G(\omega_{k_0}) > 0$.

$$0 \underset{\substack{\uparrow \\ Q \in M}}{=} E_Q[G^*] = \sum_{k=1}^{K} \underbrace{G^*(\omega_k)}_{\geq 0} \underbrace{Q_k}_{> 0} \geq G(\omega_{k_0}) Q_{k_0} > G(\omega_{k_0}) > 0$$

!!!

Hence $W \cap A = \emptyset$

$\Rightarrow$) Suppose that $W \cap A = \emptyset$. Consider the set

$$A^+ = \{ X \in A : E[X] = 1 \}$$

which is closed, bounded and convex subset of $\mathbb{R}^K$.

Note that $A^+ \subset A$ and, since $W \cap A = \emptyset$, we also have that $A^+ \cap W = \emptyset$.

Applying Corollary 4.11 with $V = W$ and $K = A^+$ we have that $\exists$ $Y \in \mathbb{R}^K$ such that

$$Y^T G^* = 0 \ , \quad G^* \in W \quad (\text{i.e. } Y \in W^\perp)$$

and $$Y^T X > 0 \qquad \forall X \in A^+ \qquad (***)$$

Note that $X = (0, \dots, 0, \underbrace{\frac{1}{p_k}}_{>0}, 0, \dots, 0)^T \in A^+$ and $(***)$ implies that

$$Y^T X = \frac{Y_k}{p_k} > 0 \iff Y_k > 0$$

Therefore $Y > 0$ and we can define its normalised version

$$Q_K := \frac{Y_k}{\sum_{j=1}^{K} Y_j} > 0 \ , \ k = 1, \dots K \ , \quad \sum_{k=1}^{K} Q_k = 1$$

Hence, $Q \in P^+$. Moreover, since $W^\perp$ is a linear subspace and $Y \in W^\perp$, we have that $Q = \left(\sum_{k=1}^{K} Y_k\right)^{-1} Y \in W^\perp$

Therefore, we have proved that $Q \in W^\perp \cap P^+ \Rightarrow \underset{\overset{\parallel}{M}}{W^\perp \cap P^+} \neq \emptyset$.

Example 3)

$B(0) = 1$, $B(1) = \frac{10}{9}$, $r = \frac{1}{9}$, $K=3$, $N=2$

$S_1(0) = 5$, $S_2(0) = 10$

$$S_1^*(1,\omega) = \begin{cases} 6 & \omega = \omega_1 \\ 6 & \omega = \omega_2 \\ 4 & \omega = \omega_3 \end{cases} \quad \text{and} \quad S_2^*(1,\omega) = \begin{cases} 12 & \omega = \omega_1 \\ 8 & \omega = \omega_2 \\ 8 & \omega = \omega_3 \end{cases}$$

We look for $Q = (Q_1, Q_2, Q_3)^T$ s.t.

$$\begin{cases} E_Q[S_1^*(1)] = S_1^*(0) = 5 \\ E_Q[S_2^*(1)] = S_2^*(0) = 10 \end{cases} \iff \begin{cases} 6Q_1 + 6Q_2 + 4Q_3 = 5 \\ 12Q_1 + 8Q_2 + 8Q_3 = 10 \end{cases} \quad (1)$$

Q must also satisfy $Q_1 + Q_2 + Q_3 = 1$ (2)

$Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$ (3)

The equations (1) and (2) have the unique solution

$Q = (1/2, 0, 1/2)^T$ but this Q does not satisfy (3)

Therefore, there are no RNPM in this market

By the FFTAP there must exist AO.

Let's find an AO. First note that

$$\Delta S_1^*(\omega) = \begin{cases} 1 & \omega = \omega_1 \\ 1 & \omega = \omega_2 \\ -1 & \omega = \omega_3 \end{cases} \quad \text{and} \quad \Delta S_2^*(\omega) = \begin{cases} 2 & \omega = \omega_1 \\ -2 & \omega = \omega_2 \\ -2 & \omega = \omega_3 \end{cases}$$

An AO is an element in $W \cap A$, so we need to characterize $W \cap A$.

$$W = \left\{ X \in \mathbb{R}^3 : X - G^* = H_1 \Delta S_1^* + H_2 \Delta S_2^* = \begin{pmatrix} H_1 + 2H_2 \\ H_1 - 2H_2 \\ -H_1 - 2H_2 \end{pmatrix} \text{ for some } H_1, H_2 \in \mathbb{R} \right\}$$

$$= \left\{ X \in \mathbb{R}^3 : \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & -2 \\ -1 & -2 \end{pmatrix}}_{\Delta S^* \in \mathbb{R}^{3\times 2}} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad H \in \mathbb{R}^2 \right\}$$

Note that ΔS_1^* and ΔS_2^* are 2 independent vectors in $\mathbb{R}^3$ and they form a basis for W, which is a subspace of $\mathbb{R}^3$ with $\dim(W) = 2$.

Therefore, W is hyperplane of $\mathbb{R}^3$ containing 0.

Here the vectors in W satisfy

$$
\begin{aligned}
0 = a^T X &= a_1 X_1 + a_2 X_2 + a_3 X_3 \quad (\text{for some } a \neq (0,0,0)) \\
&= a_1 (H_1 + 2 H_2) + a_2 (H_1 - 2 H_2) + a_3 (-H_1 - 2 H_2) \\
&= (a_1 + a_2 - a_3) H_1 + (2a_1 - 2a_2 - 2a_3) H_2
\end{aligned}
$$

$$
\iff \left. \begin{aligned} a_1 + a_2 - a_3 &= 0 \\ 2a_1 - 2a_2 - 2a_3 &= 0 \end{aligned} \right\}
$$

Taking $a_2 = 0$ we get that $a_1 = a_3$ and we can consider $a = (1, 0, 1)^T$

Hence,

$$W = \{ X \in \mathbb{R}^3 : a^T X = X_1 + X_3 = 0 \}$$ and

$$W^\perp = \{ X \in \mathbb{R}^3 : X = \lambda a = (\lambda, 0, \lambda)^T, \lambda \in \mathbb{R} \}$$

Recall that $A = \{ X \in \mathbb{R}^3 : X \geq 0, X \neq 0 \}$. Then, it follows that $W \cap A = \{ X \in \mathbb{R}^3 : X = (0, X_2, 0)^T, X_2 > 0 \}$

Fix $x_2 > 0$. Let's find H_1 and H_2 such that at time $t = 1$ we have $V^*(1) = (0, x_2, 0)^T \in W \cap A$.

We can solve

$$\left.\begin{aligned} H_1 + 2H_2 &= 0 \\ H_1 - 2H_2 &= x_2 \end{aligned}\right\} \Rightarrow H_1 = \frac{x_2}{2}, \quad H_2 = -\frac{x_2}{4}$$

Setting

$$H_0 = -H_1 S_1^*(0) - H_2 S_2^*(0) = -\frac{x_2}{2} 5 + \frac{x_2}{4} 10 = 0$$

we get that $V^*(0) = 0$ (we already knew that).

$H = \left(0, \frac{x_2}{2}, -\frac{x_2}{4}\right)^T$ is an AO for every $x_2 > 0$.