3. Basic Financial Derivatives

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Outline

Introduction

Introduction

- The valuation of financial derivatives will be based on the principle of no arbitrage.

Definition 1

Arbitrage means making of a guaranteed risk free profit with a trade or a series of trades in the market.

Definition 2

An ***arbitrage free market*** is a market which has no opportunities for risk free profit.

Definition 3

An ***arbitrage free price*** for a security is a price that ensure that no arbitrage opportunity can be designed with that security.

- The principle of no arbitrage states that the markets must be arbitrage free.
- Some financial jargon will be used in what follows.

Financial jargon

- One says that has/takes a ***long position*** on an asset if one owns/is going to own a positive amount of that asset.
- One says that has/takes a ***short position*** on an asset if one has/is going to have a negative amount of that asset.
- Being short on money means borrowing. You can take a short position on many financial assets by ***short selling***.

Example 4 (Short selling)

- To implement some trading strategy you need to sell some amount of shares (to get money and invest in other assets).
- The problem is that you do not have any shares right now.
- You can borrow the shares from another investor for a time period (paying interest) and sell the borrowed shares in the market.
- At the end of the borrowing period you must buy again the shares in the market and give them back to the lender.

Forward Contracts

Forward contracts

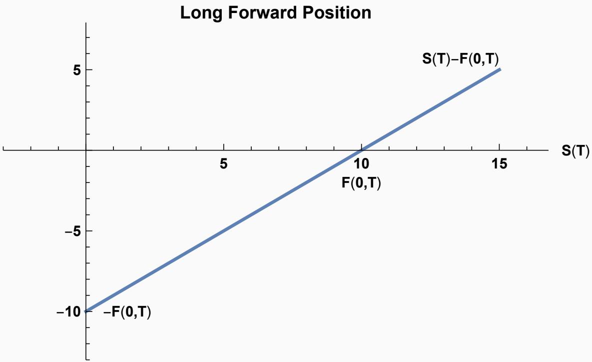
Definition 5

A ***forward contract*** is an agreement to buy or sell an asset on a fixed date in the future, called the ***delivery time***, for a price specified in advance, called the ***forward price***.

- The party selling the asset is said to be taking a ***short forward position*** and the party buying the asset is said to be taking a ***long forward position***.
- Both parties are obliged to fulfill the terms of the contract.
- The main reason to enter into a forward contract agreement is to become independent of the unknown future price of a risky asset.
- Assume that the contract is entered at time $t = 0$, the delivery time is $t = T$ and denote by $F(0, T)$ the forward price.
- The time t price of the underlying asset is denoted $S(t)$.
- Due to the symmetry of the contract, no payment is made by either party at the beginning of the contract, $t = 0$.

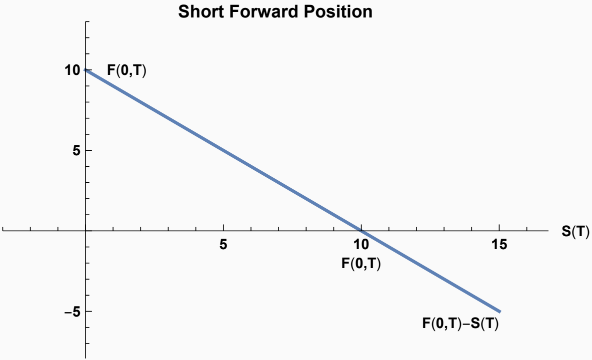
Forward contracts

- At delivery, the party with the long position makes a profit if $F(0,T) < S(T)$, while the party with the short position will make a loss (exactly of the same magnitude as the profit of the other party).
- If $F(0,T) > S(T)$ the situation is reversed.
- The payoff at delivery for a long forward position is:



Forward contracts

- The payoff at delivery for a short forward position is:



- If the contract is initiated at $t < T$, we will write $F(t,T)$ for the forward price and the payoffs will be $S(T) - F(t,T)$ (long position) and $F(t,T) - S(T)$ (short position).

Outline

Forward price

- As no payment is made at the beginning of the forward contract, the main problem is to find the forward price $F(0,T)$ such that both parties are willing to enter into such agreement.
- One possible approach would be to compute the present value, which we know that is zero, by discounting the expected payoff of the contract.
- That is,
$$0 = V(0) = e^{-rT}\mathbb{E}\left[S(T) - F(0,T)\right],$$
which yields $F(0,T) = \mathbb{E}\left[S(T)\right]$.
- Note that $F(0,T)$ would depend on the distribution of $S(T)$, hence, we would only have translated the problem to agree on which distribution use.

Forward price

- The solution comes from the fact that we can also invest in the money market and there exists only one value for $F(0,T)$ such avoid arbitrages.
- The price obtained does not depend on the distribution of $S(T)$.

Remark 6 (Buy and hold strategy)

- Borrow $S(0)$ NOK, to buy the asset at time zero
- Hold it until time T.
- At time T, the amount $S(0)\,e^{rT}$ to be paid to settle the loan is a natural candidate for $F(0,T)$.

Forward price

Theorem 7

The forward price $F(0,T)$ *is given by*

$$F(0,T) = S(0)\, e^{rT}, \tag{1}$$

where r *is the constant risk free interest rate under continuous compounding. If the contract is initiated at time* $t \leq T$*, then*

$$F(t,T) = S(t)\, e^{r(T-t)}. \tag{2}$$

Proof.

Smartboard □

Forward price

Remark 8

- The formula in the previous theorem applies as long as the underlying asset does not generate an income (dividends) or a cost (storage and insurance costs for commodities).
- In this lecture we will, many times, be implicitly assuming that the underlying is a stock which does not pay dividends.

Remark 9

In the case considered here we always have

$$F(t,T) = S(t)\, e^{r(T-t)} > S(t)\,.$$

Moreover, the difference $F(t,T) - S(t)$, called the basis, converges to 0 as t converges to T.

Outline

Value of a forward contract

- Every forward contract has value zero when initiated.
- As time goes by, the price of the underlying asset may change and, along with it, the value of the forward contract.

Theorem 10

The time t value of a long forward position with forward price $F(0,T)$ is given by

$$V(t) = \left(F(t,T) - F(0,T)\right) e^{-r(T-t)}, \qquad 0 \le t \le T, \tag{3}$$

where $F(t,T)$ is the forward price of a contract starting at t and with delivery date T.

Proof.

Smartboard. □

Value of a forward contract with forward price K

Remark 11

- Consider a forward contract with forward price K instead of $F(0,T)$.
- The value of this contract at time t will be given by equation (3) in Theorem 10 with $F(0,T)$ replaced by K,

$$V_K(t) = (F(t,T) - K)\,e^{-r(T-t)}. \tag{4}$$

- Such contract may have non-zero initial value

$$V_K(0) = (F(0,T) - K)\,e^{-rT} = S(0) - Ke^{-rT}. \tag{5}$$

Futures

Futures

- One of the two parties in a forward contract will lose money.
- There is a risk of default (not being able to fulfill the contract) by the party losing money.
- Futures contracts are designed to eliminate such risk.

Definition 12

A ***futures contract*** is an exchange-traded standardized agreement between two parties to buy or sell an asset at an specified future time and at a price agreed today. The contract is marked to market daily and guaranteed by a clearing house.

- Assume that time is discrete with steps of length τ, typically one day.
- The market dictates the so called ***futures prices*** $f\,(n\tau, T)$ for each time step $\{n\tau\}_{n\geq 0}$ such that $n\tau \leq T$.
- These prices are random (and not known at time 0) except for $f\,(0, T)$.

Futures

- As in the case of forward contracts, it costs nothing to take a futures position.
- However, a futures contract involves a random cash flow, known as ***marking to market***.
- Namely, at each time step $n\tau \leq T$,$n \geq 1$, the holder of a long futures position will receive the amount $f(n\tau, T) - f\left((n-1)\tau, T\right)$ if positive, or he/she will have to pay it if negative.
- The opposite payments apply for a short futures position.

The following conditions are imposed:

1. The futures price at delivery is $f\left(T, T\right) = S\left(T\right).$
2. At each $n\tau \leq T$,$n \geq 1$, the value of a futures position is reset to zero, after marking to market.

In particular 2. implies that it costs nothing to close, open or modify a futures position at any time step between 0 and T.

Remark 13

- Each investor entering into a futures contract has to set up a deposit, called the ***initial margin***, which is kept by the clearing house as collateral.
- In the case of a long futures position the amount is added/subtracted to the deposit, depending if it is positive/negative, at each time step. The opposite amount is added/subtracted for a short position.
- Any excess above the initial margin can be withdrawn.
- However, if the deposit drops below a certain level, called the ***maintenance margin***, the clearing house will issue a ***margin call***, requesting the investor to make a payment and restore the deposit to the level of the initial margin.
- If the investor fails to respond to a margin call the futures position is closed by the clearing house.

Remark 14

- Futures markets are very liquid (high number of transactions) due to standardization and the presence of a clearing house.
- There is no risk of default, in contrast to forward contracts negotiated directly between two parties.
- A negative aspect of of the highly standardized contracts is that you may not find a contract that actually covers the risk you want to hedge.
- For example, you want protection against adverse movements of the share prices of a company, but in the market there are no futures on the share price of that company.
- In this case, you may use futures on a stock index containing the company, but the hedge is less perfect and more risky.

Futures

Theorem 15

If the interest rate r is constant, then $f(0,T) = F(0,T)$.

Proof.

Smartboard. $\square$

- If we assume that the interest rates change unpredictably the strategy II (in the proof of Theorem 15) cannot be implemented.
- Hence, in an economy with constant interest rate r we obtain a simple structure of futures prices

$$f(t,T) = S(t)\, e^{r(T-t)}. \tag{6}$$

- Note that the futures are random but only due to $S(t)$.

Options

Options

Definition 16

A ***European call/put option*** is a contract giving the holder the right (but not the obligation) to buy/sell an asset for a price K fixed in advance, known as the ***exercise price*** or ***strike price***, at a specified future date T, called the ***exercise time*** or ***expiry date***.

Definition 17

An ***American call/put option*** is a contract giving the holder the right (but not the obligation) to buy/sell an asset for a strike price K, also fixed in advance, at any time between now and the expiry date T.

- Some underlying assets may be impossible to buy or sell (e.g., stock indices).
- The option then is cleared in cash.

Options

- An option is determined by its ***payoff***, which for a European call is

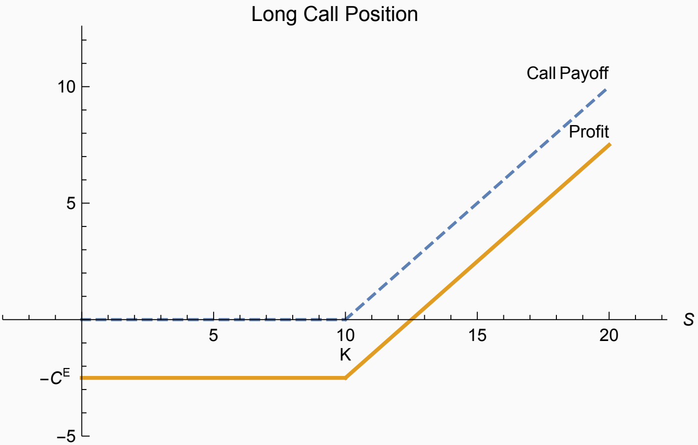
$$(S(T)-K)^{+}=\max(0,S(T)-K)=\begin{cases} S(T)-K & if \quad S(T)>K \\ 0 & if \quad S(T)\leq K \end{cases},$$

  and for a put option is $(K-S(T))^{+}$.
- Since the payoffs are non negative, a premium must be paid to buy an option, otherwise there is an arbitrage opportunity.
- The ***premium*** is the market price of the option at time 0.

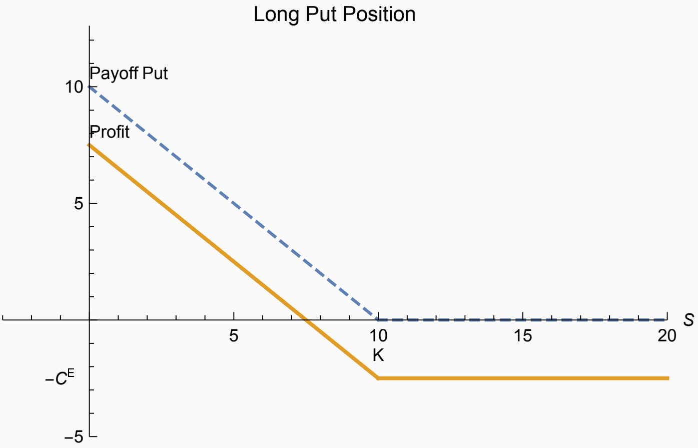
Options

- The prices of European calls and puts will be denoted by C^E and P^E.
- We will use the notation C^A and P^A for the American ones.
- The same constant interest rate r will apply for borrowing and lending money and we will use continuous compounding.
- The ***gain/profit of an option*** buyer (seller, also known as writer) is the payoff minus (plus) the premium C^E or P^E paid (received) for the option.

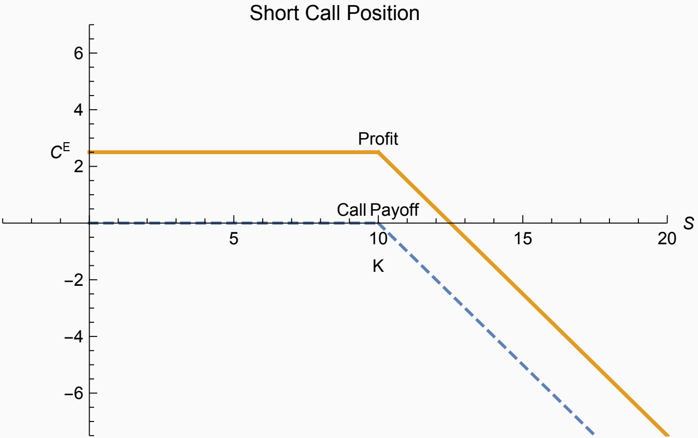
Options



Options

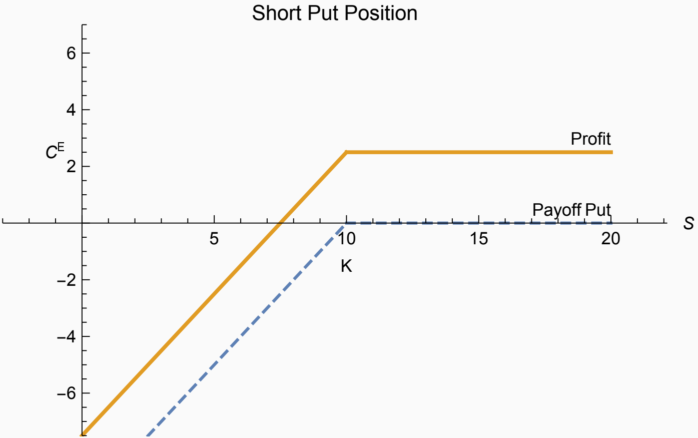


Options



Short Call Position
6
4
2
−2
−4
−6
C^E
Profit
Call Payoff
5
10
15
20
S
K

Options



Outline

Put-call parity

- In what follows we will find bounds and some general properties for option prices.
- We will use the principle of no arbitrage alone, without any assumption on the evolution of the underlying asset prices.
- Take a long position with a European call and a short position with a European put, both with the same strike K and expiry time T.
- We obtain a portfolio having the same payoff as a long forward position with forward price K and delivery time T, that is,

$$(S(T) - K)^{+} - (K - S(T))^{+} = S(T) - K.$$

- As a result, the present value of such portfolio of options should be that of a forward contract with forward price K, which is

$$S(0) - Ke^{-rT},$$

according to Remark 11.

Put-call parity

Theorem 18 (European put-call parity)

For a stock paying no dividends, the prices of European call and put options, both with the same strike price K and exercise time T, satisfy

$$C^E - P^E = S(0) - Ke^{-rT}. \tag{7}$$

Proof.

Smartboard. □

Theorem 19 (American put-call parity estimates)

For a stock paying no dividends, the price of American call and put options, both with the same strike price K and expiry time T, satisfy

$$C^A - P^A \leq S(0) - Ke^{-rT}, \tag{8}$$

$$C^A - P^A \geq S(0) - K. \tag{9}$$

Proof.

Smartboard □

Outline

Bounds on option prices

- In this section we will assume that all the options have the same strike K and expiry time T.
- We start by noting the following obvious inequalities

$$\left.\begin{array}{l} 0 \leq C^E \leq C^A, \\ 0 \leq P^E \leq P^A. \end{array}\right\} \qquad (10)$$

- The option prices must be non-negative because they have non-negative payoff.
- American options should be more expensive because they give at least the same rights as their European counterparts.

Bounds on option prices

Proposition 20

On a stock paying no dividends one has that

$$\left(S(0) - Ke^{-rT}\right)^{+} = \max\left(0, S(0) - Ke^{-rT}\right) \leq C^{E} < S(0), \quad (11)$$

$$\left(Ke^{-rT} - S(0)\right)^{+} = \max\left(0, Ke^{-rT} - S(0)\right) \leq P^{E} < Ke^{-rT}. \quad (12)$$

Proof.

Smartboard □

Theorem 21

On a stock paying no dividends one has that

$$C^{E} = C^{A}. \quad (13)$$

Proof.

Smartboard. □

Bounds on option prices

Remark 22

- As $C^A \geq C^E$ and $C^E \geq S(0) - Ke^{-rT}$, it follows that

$$C^A > S(0) - K,$$

  if $r > 0$.
- Because the price of the American option is greater than its payoff, the option will sooner be sold than exercised at time 0.
- Similar inequalities hold for $t < T$ and one can repeat the arguments to conclude that the American option will never be exercised prior to the expiry time.
- This also shows that the American option is equivalent to the European option.

Bounds on option prices

Proposition 23

On a stock paying no dividends one has that

$$\left(S(0) - Ke^{-rT}\right)^+ = \max\left(0, S(0) - Ke^{-rT}\right) \leq C^A < S(0), \tag{14}$$

$$(K - S(0))^+ = \max\left(0, Ke^{-rT} - S(0)\right) \leq P^A < K. \tag{15}$$

Proof.

Exercise. □

Outline

Variables determining option prices

- Here we will study how the option prices depend on variables such the strike K, the current price of the underlying $S(\dot{0})$, and the expiry time T.
- We shall analyse option prices as functions of one of the variables, keeping the remaining variables constant.

European options: dependence on the strike price

Proposition 24

If $K_1 < K_2$, then

1. *Monotonicity:* $C^E(K_1) > C^E(K_2)$ *and* $P^E(K_1) < P^E(K_2)$.
2. *Lipschitz continuity:*

$$C^E(K_1) - C^E(K_2) < e^{-rT}(K_2 - K_1),$$
$$P^E(K_2) - P^E(K_1) < e^{-rT}(K_2 - K_1).$$

3. *Convexity: For* $\alpha \in (0,1)$ *we have*

$$C^E(\alpha K_1 + (1-\alpha) K_2) \leq \alpha C^E(K_1) + (1-\alpha) C^E(K_2),$$
$$P^E(\alpha K_1 + (1-\alpha) K_2) \leq \alpha P^E(K_1) + (1-\alpha) P^E(K_2).$$

Proof.

Smartboard. □

European options: dependence on the underlying asset price

- The current price $S(0)$ of the underlying asset is given by the market and cannot be changed.
- But we can consider an option on a portfolio of x shares, worth $S = xS(0)$.
- The payoff of a European call with strike K on such portfolio, to be exercised at time T, will be $(xS(T) - K)^+$.
- We shall study the dependence of option prices on S.
- We will denote the call and put prices by $C^E(S)$ and $P^E(S)$.

European options: dependence on the underlying asset price

Proposition 25

If $S_1 < S_2$, then

1. *Monotonicity:* $C^E(S_1) < C^E(S_2)$ *and* $P^E(S_1) > P^E(S_2)$.
2. *Lipschitz continuity:*

$$
\begin{aligned}
C^E(S_2) - C^E(S_1) &< S_2 - S_1, \\
P^E(S_1) - P^E(S_2) &< S_2 - S_1.
\end{aligned}
$$

3. *Convexity: For* $\alpha \in (0,1)$ *we have*

$$
\begin{aligned}
C^E(\alpha S_1 + (1-\alpha) S_2) &\le \alpha C^E(S_1) + (1-\alpha) C^E(S_2), \\
P^E(\alpha S_1 + (1-\alpha) S_2) &\le \alpha P^E(S_1) + (1-\alpha) P^E(S_2).
\end{aligned}
$$

Proof.

Smartboard. □

American options: dependence on the strike price

Proposition 26

If $K_1 < K_2$*, then*

1. *Monotonicity:* $C^A(K_1) > C^A(K_2)$ *and* $P^A(K_1) < P^A(K_2)$.
2. *Lipschitz continuity:*

$$\begin{aligned} C^A(K_1) - C^A(K_2) &< (K_2 - K_1), \\ P^A(K_2) - P^A(K_1) &< (K_2 - K_1). \end{aligned} \tag{16}$$

3. *Convexity: For* $\alpha \in (0,1)$ *we have*

$$\begin{aligned} C^A(\alpha K_1 + (1-\alpha) K_2) &\leq \alpha C^A(K_1) + (1-\alpha) C^A(K_2), \\ P^A(\alpha K_1 + (1-\alpha) K_2) &\leq \alpha P^A(K_1) + (1-\alpha) P^A(K_2). \end{aligned}$$

Proof.

Smartboard. □

American options: dependence on the underlying asset price

- As in the European case, we consider options on a portfolio of x shares worth $S = xS(0)$.

Proposition 27

If $S_1 < S_2$ *then*

1. *Monotonicity:* $C^A(S_1) < C^A(S_2)$ *and* $P^A(S_1) > P^A(S_2)$.
2. *Lipschitz continuity:*

$$C^A(S_2) - C^A(S_1) < S_2 - S_1,$$
$$P^A(S_1) - P^A(S_2) < S_2 - S_1.$$

3. *Convexity: For* $\alpha \in (0,1)$ *we have*

$$C^E(\alpha S_1 + (1-\alpha) S_2) \leq \alpha C^E(S_1) + (1-\alpha) C^E(S_2),$$
$$P^E(\alpha S_1 + (1-\alpha) S_2) \leq \alpha P^E(S_1) + (1-\alpha) P^E(S_2).$$

American options: dependence on expiry time

Proposition 28

If $T_1 < T_2$, *then* $C^A(T_1) \leq C^A(T_2)$ *and* $P^A(T_1) \leq P^A(T_2)$.

Proof: (only for calls, the proof for puts being analogous).

Suppose that $C^A(T_1) > C^A(T_2)$, then

- Sell the option with shorter time to expiry and buy the one with longer time to expiry, investing the balance risk free.
  - If the option sold is exercised at time $t \leq T_1$, we can exercise the other option to cover our liability.
  - The risk-less profit will be $\left(C^A(T_1) - C^A(T_2)\right) e^{rt} > 0$.

$\square$

Remark 29

The previous arguments do not work for European options because early exercise is not possible.

Outline

Time value of options

Definition 30

We say that at time $0 \leq t \leq T$ a call option with strike K is

- ***(deep) in the money*** if $S(t) \overset{(\gg)}{>} K$,
- ***at the money*** if $S(t) = K$,
- ***(deep) out of the money*** if $S(t) \overset{(\ll)}{<} K$.

The same terminology applies to put options but with the inequalities reversed.

Definition 31

At time $0 \leq t \leq T$, the ***intrinsic value of a call (put) option*** with strike K is equal to $(S(t) - K)^{+}$ $((K - S(t))^{+})$.

Time value of options

Remark 32

- The intrinsic value of out of the money or at the money options is zero.
- Options in the money have positive intrinsic value.
- The price of an American option prior to expiry must be greater than its intrinsic value.
- The price of a European option prior to expiry may be greater or smaller than its intrinsic value.

Time value of options

Definition 33

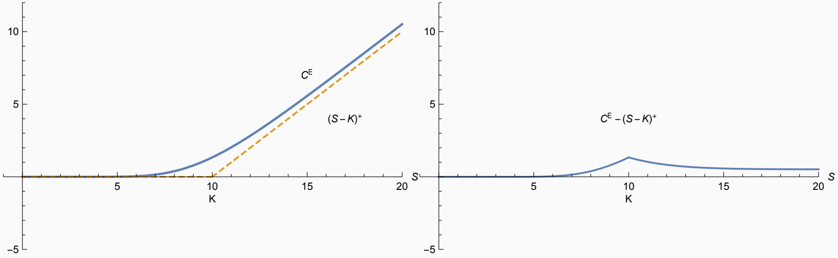
The ***time value of an option*** is the difference between the price of the option and its intrinsic value, that is,

$$\begin{aligned}
&C^E(t) - (S(t) - K)^+, && \text{European call}\\
&P^E(t) - (K - S(t))^+, && \text{European put}\\
&C^A(t) - (S(t) - K)^+, && \text{American call}\\
&P^A(t) - (K - S(t))^+. && \text{American put.}
\end{aligned}$$

Here, the argument t in the option prices denotes the current time and NOT the expiry time as in Proposition 28.

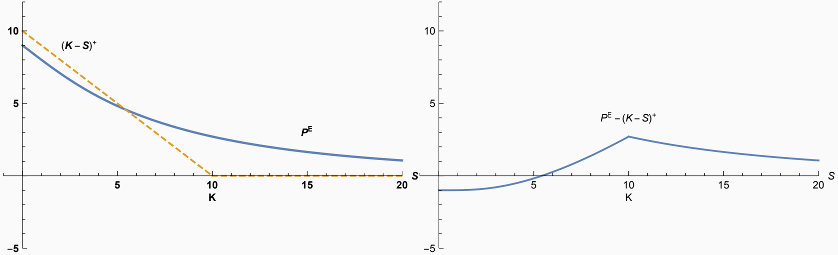
Time value of options

- The time value of a European call as a function of $S(t)$ is always nonnegative.
- For in the money calls, the time value is bigger than $K - Ke^{-r(T-t)}$, due to the inequality $C^E(t) \geq S(t) - Ke^{-r(T-t)}$ (this inequality is proved using analogous arguments as those in the proof of inequality (11) in Proposition 20).
- The same applies to an American call because their prices coincide.



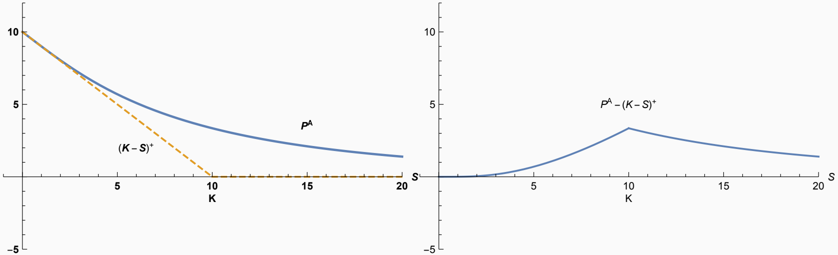
Time value of options

- The time value of a European put may be negative.
- This happens if the put option is deep in the money, because we can only exercise the option at time T and there is a considerable risk that in the meanwhile the stock price rises.



Time value of options

- The time value of an American put is always nonnegative.



Time value of options

Proposition 34

For any European or American call or put with strike price K, the time value attains its maximum at $S = K$.

Proof. (only for European calls).

If $S \leq K$ the intrinsic value is zero. Since $C^E(S)$ is an increasing function of S, this means that the time value of the call is increasing for $S \leq K$. If $K \leq S_1 < S_2$, we have that $C^E(S_2) - C^E(S_1) \leq S_2 - S_1$ and, hence,

$$
\begin{aligned}
C^E(S_2) - S_2 &\leq C^E(S_1) - S_1 \\
&\Updownarrow \\
C^E(S_2) - (S_2 - K)^+ &= C^E(S_2) - S_2 + K \\
&\leq C^E(S_1) - S_1 + K = C^E(S_1) - (S_1 - K)^+,
\end{aligned}
$$

which yields that the time value of the call is a decreasing function of S if $S \geq K$. Therefore, the maximum is at $S = K$. □

Outline

Hedging and speculating with options

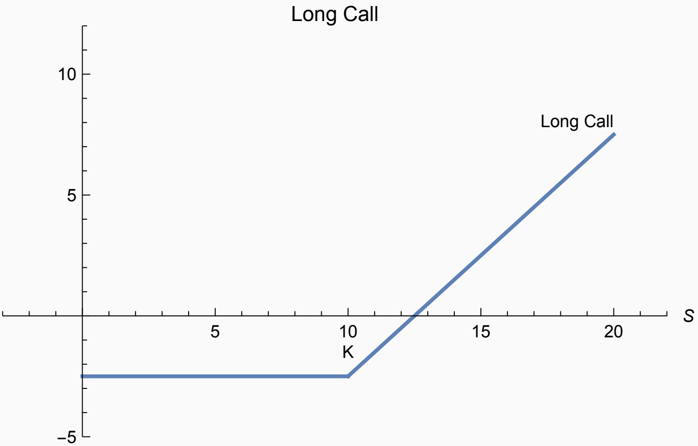
- Suppose that you are an investor with specific views on the future behavior of stock prices and you are willing to take/avoid risks.
- Using European options, we will show some of the most popular strategies, used to hedge/speculate on stock prices.
- These strategies can be classified into three categories:
  - ***Bullish***: These strategies are used when a rise in the price of the stock is expected.
  - ***Bearish***: These strategies are used when a fall in the prices of the stock is expected.
  - ***Neutral or non-directional***: These strategies are used when no clear direction in the price of the stock is expected. They bet on the volatility (standard deviation) of the stock price.

Hedging and speculating with options

- We will assume that all the options are on the same stock.
- We will also assume have the same strike K and expiry time T, unless stated otherwise.
- By building portfolios of calls, puts, underlying and bonds, you can replicate any piecewise linear terminal payoff function.
- Thanks to the put-call parity different combinations of calls, puts, stocks and risk free investment may produce the same terminal profits.
- Hence, there is no unique way of implementing the following strategies.
- We will use a solid line to plot the total profit and dashed lines to plot individual option profits.

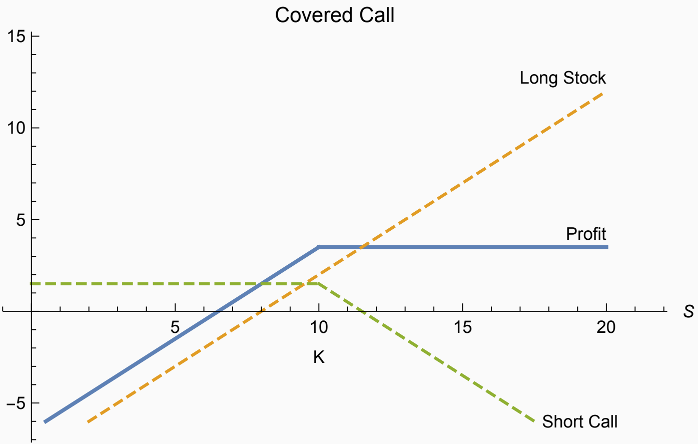
Bullish strategies: long call

- Buy a call option.
- You expect a high rise in the price of the stock.
- Loses are limited.



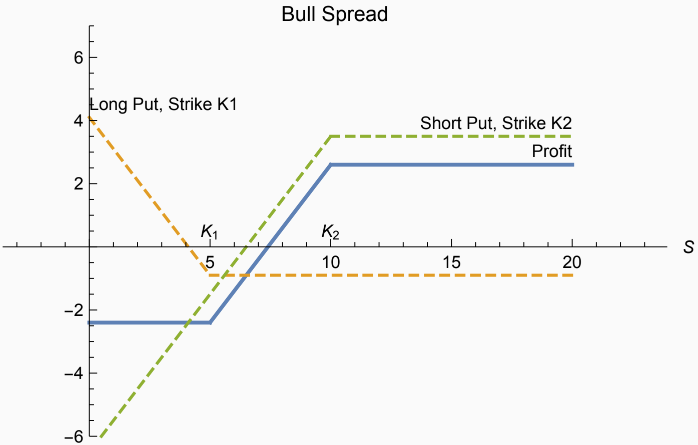
Bullish strategies: covered call

- Sell a call option and buy the stock.
- You expect a moderate rise in the price of the stock.
- Loses can be very high.



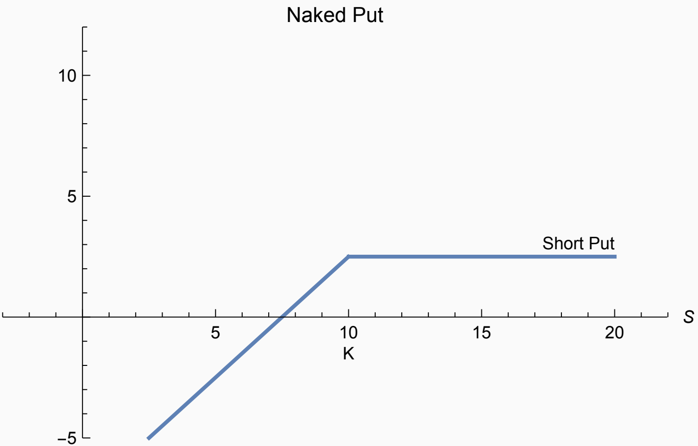
Bullish strategies: bull spread

- Buy a put option with strike K_1 and sell a put option with higher strike K_2.
- You expect a moderate rise in the price of the stock.
- Loses are limited.



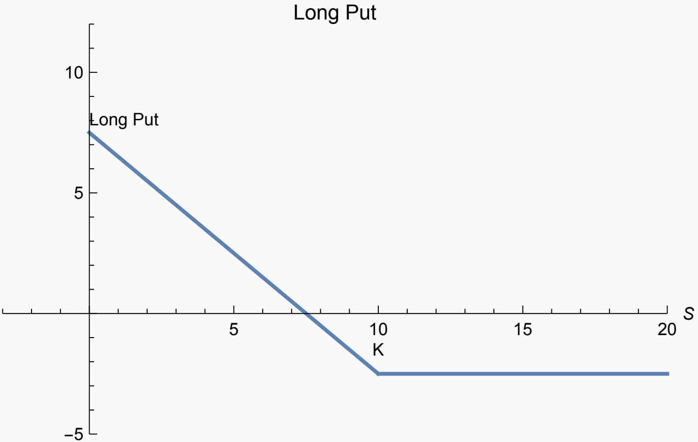
Bullish strategies: naked put

- You sell a put option with strike K.
- You expect that at expiry time S will be above K .
- Loses can be very high.



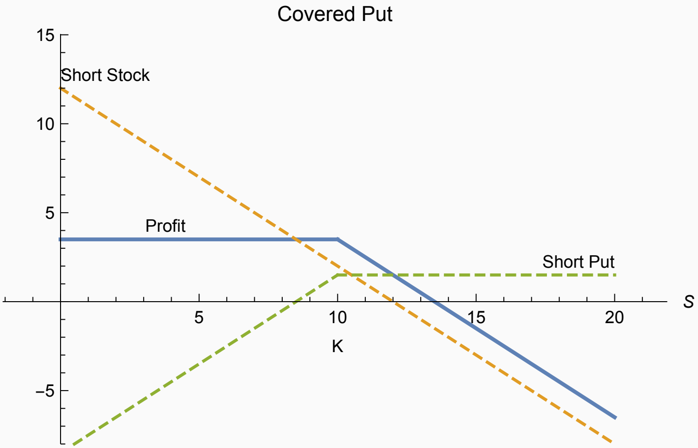
Bearish strategies: long put

- You buy a put option with strike K.
- You expect a big drop in the price of the stock.
- Loses are limited.



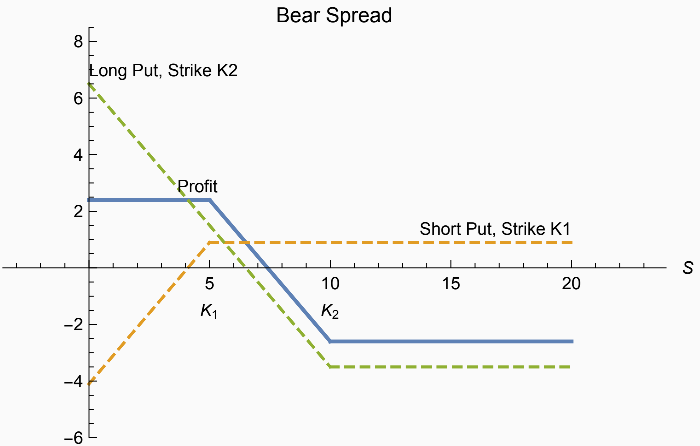
Bearish strategies: covered put

- You sell a put option and sell the stock.
- You expect a moderate drop in the price of the stock.
- Loses can be very high.



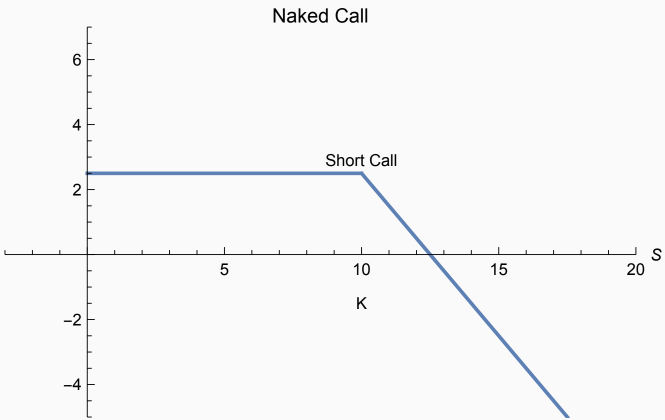
Bearish strategies: bear spread

- You buy a put option with strike K_2 and sell a put option with lower strike K_1.
- You expect a moderate drop in the price of the stock.
- Loses are limited.



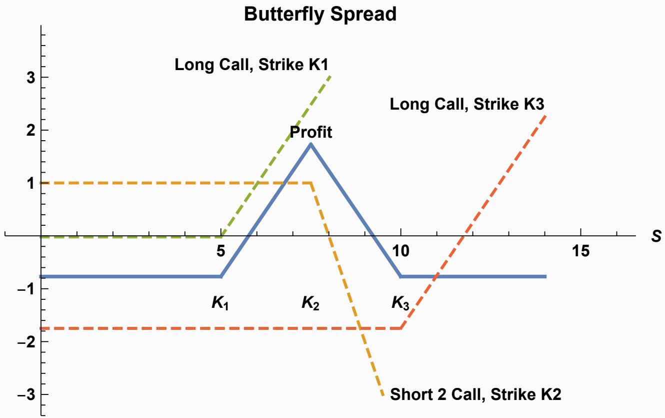
Bearish strategies: naked call

- You sell a call option with strike K.
- You expect that at expiry time S will be below K .
- Loses can be very high. (Unbounded, in theory)



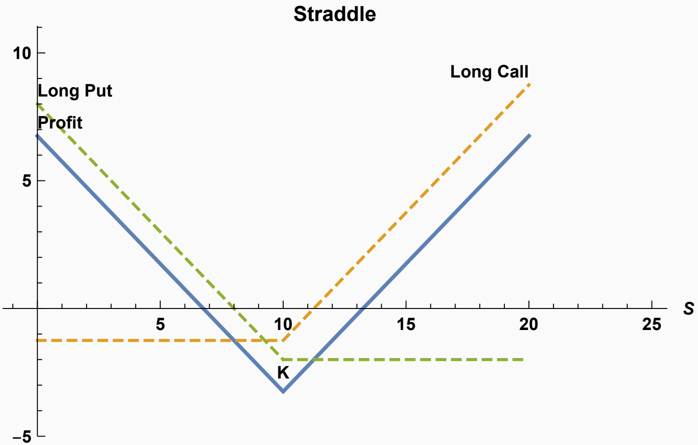
Neutral strategies: butterfly spreads

- Let $0 < K_1 < K_2 < K_3$. Buy a call option with strike K_1 and a call option with strike K_3 and sell two call options with strike K_2.
- You make a profit if the S stays close to K_2.
- Hence, you expect a low volatility for the stock price.
- Loses are limited.



Neutral strategies: straddles

- You buy a call option and a put option with the same strike K.
- You make a profit if S ends far away from K.
- Hence, you expect a high volatility for the stock price.
- Loses are limited.



Neutral strategies: strangles

- Let $0 < K_1 < K_2$. Buy a call option with strike K_2 and a put option with strike K_1.
- You make a profit if S ends far out of the interval $[K_1, K_2]$.
- Hence, you expect a very high volatility for the stock price.
- Loses are limited. Strangles are cheaper than straddles.