4. Review of Linear Programming

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Outline

Linear Programming

Linear programming

- Linear programming $(\mathbf{LP})$ is about solving optimization problems where the objective function and the constraints are linear.
- The optimization problem can be finding a maximum or a minimum and the constraints can be given by equalities and/or inequalities.
- In what follows most inequalities will be vector inequalities, that is, the inequalities hold componentwise.

Linear programming

- All $\mathbf{LP}$ problems can be written in the following standard form

Primal Problem $(\mathbf{P})$

$$\max J(x) = \max c^T x$$
$$\text{subject to } Ax \leq b,$$
$$x \geq 0,$$

where $x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

Linear programming

We will use the following notation:

- **Objective function**: It is the function J to be optimized. In this case the linear function $J(x) = c^T x$.
- **Feasible set/solution**: $x \in \mathbb{R}^n$ is a feasible solution if satisfies the constraints, i.e., $Ax \leq b, x \geq 0$. The feasible set F_P is the convex set defined by all feasible solutions, i.e,

$$F_P := \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}.$$

- **Optimal solution**: $\hat{x} \in F_P$ such that

$$J(\hat{x}) = c^T \hat{x} = \max \left\{ c^T x : Ax \leq b, x \geq 0 \right\}.$$

- **Optimal value**: It is the value (finite) of the objective function at an optimal solution , i.e., $J(\hat{x})$.

Linear programming

There are three different cases regarding the problem $\mathbf{P}$:

1. There exists an optimal solution (or many) and only one optimal value.
2. $F_P = \emptyset$, then the optimal value is set to $-\infty$. We say that the problem is ***not feasible***.
3. The problem is ***unbounded***. There exists a sequence $\{x_k\}_{k\geq 1} \subseteq F_P$ such that $J(x_k) \to_{k\to\infty} \infty$.

Reduction to the Standard Form

Reduction to the standard form

We have the following rules:

- "min"$\longrightarrow$"max": $\min J(x) = -\max J(-x)$.
- "$\geq$"$\longrightarrow$"$\leq$": Multiply the equation by -1.
- "$=$"$\longrightarrow$"$\leq$": Write as two inequalities using "$\leq$" and "$\geq$". Then apply the previous point to the inequality with "$\geq$".
- "Free variables"$\longrightarrow$"Restricted variables": Write $x = x^+ - x^-$, where $x^+ = \max(0, x) \geq 0$ and $x^- = -\min(0, x) \geq 0$ and rewrite the other constraints and the objective function in terms of x^+ and x^-.

Reduction to the standard form

- A general (iterative) method to solve $\mathbf{LP}$ problems is the simplex method (Dantzig, 1947).
- In the simplex method the constraints must be in equality form.
- We can go from "$\leq$" to "$=$" by introducing the so called slack variables $w := b - Ax$, then the problem $\mathbf{P}$ can be written as

$$\begin{aligned} & \max J(x) \\ \text{subject to } & w = b - Ax, \\ & w \geq 0, \\ & x \geq 0. \end{aligned}$$

Reduction to the standard form

Example 1

- Consider the **LP** problem

$$\begin{aligned} \max J(x) &= 3x_1 + 2x_2 \\ \text{subject to } -x_1 + 3x_2 &\le 12, \\ x_1 + x_2 &\le 10, \\ 2x_1 - x_2 &\le 10 \\ x_1 \ge 0, \quad x_2 &\ge 0 \end{aligned}$$

- This example is discussed on the smartboard.

Duality

Dual problem

- The previous example justifies the introduction of the ***dual problem*** of a LP.

Definition 2

Given the LP problem P we define its dual D as

Dual Problem (D)

$$\min J(y) = \min b^T y$$
$$\text{subject to } A^T y \geq c,$$
$$y \geq 0,$$

where $y \in \mathbb{R}^m, b \in \mathbb{R}^m, A^T \in \mathbb{R}^{n\times m}$ and $c \in \mathbb{R}^n$.

Dual problem

Remark 3

We have that

- The dual problem of a $\mathbf{LP}$ problem is also a $\mathbf{LP}$ problem.
- The dual problem provides upper bounds for the optimal value of the primal problem.
- $\mathbf{D}$ is sometimes easier to solve than $\mathbf{P}$.
- Good implementations of the simplex algorithm solve simultaneously $\mathbf{P}$ and $\mathbf{D}$.

Dual problem

Lemma 4

The dual of $\mathbf{D}$ *is* $\mathbf{P}$*.*

Proof.

We can write

$$\min\left\{b^T y : A^T y \geq c, y \geq 0\right\}$$
$$= -\max\left\{(-b)^T y : -A^T y \leq -c, y \geq 0\right\}.$$

The problem on the right hand side of the previous equation is in standard form, so we can take its dual to get

$$-\min\left\{(-c)^T x : -\left(A^T\right)^T x \geq -b, x \geq 0\right\},$$

which in standard form is $\max\left\{c^T x : Ax \leq b, x \geq 0\right\}$. □

Dual problem

- Sometimes it is convenient to find the dual of a $\mathbf{LP}$ problem without finding first its standard form.
- We assume that we have a $\mathbf{LP}$ problem in the form of a ***generalised primal problem*** $\mathbf{P_g}$
- This means that we have a primal problem with some constraints that are equalities and only R variables are restricted.

Dual problem

- That is,

Generalized Primal Problem $(\mathbf{P_g})$

$$\begin{aligned} \max J(x) &= \max c^T x \\ \text{subject to } \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \qquad i \in I, \\ \sum_{j=1}^{n} a_{ij} x_j &= b_i, \qquad i \in E, \\ x_j &\geq 0, \qquad j \in R \end{aligned}$$

where $x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $R \subseteq \{1, \ldots, n\}, I, E \subseteq \{1, \ldots, m\}$, $I \cap E = \emptyset$, and $I \cup E = \{1, \ldots, m\}$.

Dual problem

- Using the following primal-dual correspondence

	$(\mathbf{P_g})$	$(\mathbf{D_g})$	
I	Inequality constraints	Restricted variables	R
E	Equality constraints	Free variables	F
R	Restricted variables	Inequality constraints	I
F	Free variables	Equality constraints	E

we can find its associated ***generalised dual problem*** $(\mathbf{D_g})$

Dual problem

- That is a dual problem with some equality constraints and only some variables which are restricted

Generalized Dual Problem $(\mathbf{D_g})$

$$\min J(y) = \min b^T y$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad j \in R,$$

$$\sum_{i=1}^{m} a_{ij} y_i = c_j, \qquad i \in F,$$

$$y_i \geq 0, \qquad i \in I$$

where $y \in \mathbb{R}^m, c \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$,R,$F \subseteq \{1, \ldots, n\}$,$R \cap F = \emptyset, R \cup F = \{1, \ldots, n\}$.

Dual problem

Theorem 5 (Duality)

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

1. ***(Weak duality)*** *If x is feasible for $(\mathbf{P})$ and y is feasible for $(\mathbf{D})$, then*

$$c^T x = x^T c \leq x^T \left(A^T y\right) = (Ax)^T y \leq b^T y.$$

   Moreover:

   1.1 *If $(\mathbf{P})$ is unbounded $\implies$ $(\mathbf{D})$ is not feasible.*
   1.2 *If $(\mathbf{D})$ is unbounded $\implies$ $(\mathbf{P})$ is not feasible.*
   1.3 *If $c^T \hat{x} = b^T \hat{y}$ with $\hat{x}$ feasible for $(\mathbf{P})$ and $\hat{y}$ feasible for $(\mathbf{D})$, then $\hat{x}$ must solve $(\mathbf{P})$ and $\hat{y}$ must solve $(\mathbf{D})$.*

2. ***(Strong duality)*** *If either $(\mathbf{P})$ or $(\mathbf{D})$ has a finite optimal value, then so does the other, the optimal values coincide, and optimal solutions for both $(\mathbf{P})$ and $(\mathbf{D})$ exist.*

Convex Analysis

Definition 6

A set $A \subset \mathbb{R}^n$ is ***convex*** if one has that $\lambda x + (1-\lambda)\, y \in A$, for all $x, y \in A$ and $\lambda \in (0,1)$.

Definition 7

An ***hyperplane*** with normal vector $a \neq 0 \in \mathbb{R}^n$ and level $\alpha \in \mathbb{R}$ is the set

$$H_{a,\alpha} = \left\{ x \in \mathbb{R}^n : a^T x = \alpha \right\}.$$

Every hyperplane $H_{a,\alpha}$ is the intersection of the ***halfspaces***

$$H^-_{a,\alpha} = \left\{ x \in \mathbb{R}^n : a^T x \leq \alpha \right\},$$
$$H^+_{a,\alpha} = \left\{ x \in \mathbb{R}^n : a^T x \geq \alpha \right\}.$$

Definition 8

Let S and T be two sets in $\mathbb{R}^n$. We say that $H_{a,\alpha}$ ***strongly separates*** S and T if there exists $\varepsilon > 0$ such that $S \subseteq H^-_{a,\alpha-\varepsilon}$ and $T \subseteq H^+_{a,\alpha+\varepsilon}$ or viceversa.

Theorem 9 (Strong Separating Hyperplane Theorem)

Let S and T be two disjoint, non-empty, closed, convex sets in $\mathbb{R}^n$ and one of them is compact. Then, there exists an hyperplane $H_{a,\alpha}$ that strongly separates S and T.

Corollary 10

Let S be a non-empty, closed, convex set in $\mathbb{R}^n$ and such that $0 \notin S$. Then, there exist $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}_{++}$ such that

$$a^T x \geq \alpha > 0, \qquad x \in S.$$

Proof.

Smartboard. □

Corollary 11

Let V be a linear subspace of $\mathbb{R}^n$ and let K be a non-empty, compact, convex set in $\mathbb{R}^n$, such that $K \cap V = \emptyset$. Then, there exists $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}_{++}$ such that

$$\begin{aligned} a^T x &= 0, \qquad x \in V, \\ a^T y &\geq \alpha > 0, \qquad y \in K. \end{aligned}$$

Proof.

Smartboard. □

Linear Algebra

Definition 12

Given $A \in \mathbb{R}^{m \times n}$, we can consider the following fundamental linear subspaces:

- $\operatorname{col}(A)$: The ***column space*** of A, it contains all linear combinations of the columns of A.
- $\operatorname{null}(A)$: The ***null space*** of A, it contains all solutions to the system $Ax = 0$.
- $\operatorname{col}(A^T)$: The ***row space*** of A, it contains all linear combinations of the rows of A, (or columns of A^T).
- $\operatorname{null}(A^T)$: The ***left null space*** of A, it contains all solutions to the system $A^T y = 0$.

Definition 13

The ***rank*** of A, denoted $\operatorname{rank}(A)$, is the dimension of $\operatorname{col}(A)$ or $\operatorname{col}(A^T)$.

Definition 14

Let $S \subseteq \mathbb{R}^n$. We define $S^\perp$, the ***orthogonal complement*** of S, as the set of vectors in $\mathbb{R}^n$ which are orthogonal to S, that is,

$$S^\perp := \left\{ x \in \mathbb{R}^n : x^T y = 0, \quad y \in S \right\}.$$

- It is easy to check that $S^\perp$ is a linear subspace, regardless of S being a subspace or not.
- If S is a linear subspace, then $S \cap S^\perp = \{0\}$.

Proposition 15 (Orthogonal projection)

Let $v \in \mathbb{R}^n$ and let $S \subseteq \mathbb{R}^n$ be a linear subspace. Then there exist unique $x \in S$ and $y \in S^\perp$ such that

$$v = x + y.$$

We write $\mathbb{R}^n = S \oplus S^\perp$, and we say that $\mathbb{R}^n$ is the direct sum of S and $S^\perp$.

Theorem 16 (Fundamental theorem of linear algebra)

Let $A \in \mathbb{R}^{m\times n}$*. Then* $\operatorname{col}(A)$ *is orthogonal to* $\operatorname{null}(A^T)$*, and*

$$\mathbb{R}^m = \operatorname{col}(A) \oplus \operatorname{null}\left(A^T\right).$$

Moreover, $\operatorname{col}(A^T)$ *is orthogonal to* $\operatorname{null}(A)$ *and*

$$\mathbb{R}^n = \operatorname{col}\left(A^T\right) \oplus \operatorname{null}(A).$$

Proof.

Follows from Proposition 15 and the following equalities

$$\begin{aligned}
\operatorname{col}(A)^{\perp} &= \left\{ y \in \mathbb{R}^m : y^T A x = 0, \quad x \in \mathbb{R}^n \right\} \\
&= \left\{ y \in \mathbb{R}^m : x^T \left(A^T y \right) = 0, \quad x \in \mathbb{R}^n \right\} \\
&= \left\{ y \in \mathbb{R}^m : A^T y = 0 \right\} \\
&= \operatorname{null}\left(A^T \right).
\end{aligned}$$

□

Proposition 17 (Fredholm's alternative)

For every matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$, ***exactly one*** *of the following statements is true:*

1. $Ax = b$ *has a solution* $x \in \mathbb{R}^n$.
2. *There exists* $0 \neq y \in \mathbb{R}^m$ *such that* $A^T y = 0$ *and* $y^T b \neq 0$.

Proof.

- Suppose $Ax = b$ has a solution.
    - This is equivalent to $b \in \operatorname{col}(A)$.
    - Let $y = y_c + y_n \in \mathbb{R}^m$, $y_c \in \operatorname{col}(A)$,$y_n \in \operatorname{null}\left(A^T\right)$.
    - Note that
$$A^T y = A^T y_c + A^T y_n = A^T y_c$$
and
$$y^T b = y_c^T b + y_n^T b = y_c^T b.$$
    - But then, if $A^T y = 0$ we have that
$$A^T y_c = 0 \Leftrightarrow y_c = 0 \Leftrightarrow y_c^T = 0 \Longrightarrow y_c^T b = 0,$$
which also implies that $y^T b = 0$.
    - Therefore, 2. does not hold true.

□

Proof.

- Suppose that $Ax = b$ does not have a solution.
  - Note that, in this case, $b \neq 0 \in \mathbb{R}^m$, because for $b = 0$ we always have the solution $x = 0$.
  - Moreover, this is equivalent to $b \notin \text{col}\,(A)$ (i.e., $b \in \text{null}\left(A^T\right)$).
  - Then, $A^T b = 0$ and $b^T b = \|b\|^2 \neq 0$.
  - Hence, we can take $y = b$ and we have that 2. holds true.

$\square$