5. Single Period Financial Markets

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STK-MAT 3700/4700 An Introduction to Mathematical Finance

September 20, 2021

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Outline

Model Specifications

Introduction

Single period models are

- Unrealistic (prices change almost continuously in time)
- Mathematically simple (linear algebra + discrete probability)
- Useful (easily illustrate many economic principles observed in real markets)

Model specifications

Definition 1

A single period model of financial markets is specified by the following ingredients:

1. ***Initial date*** $(t = 0)$ and a ***terminal date*** $(t = 1)$.
2. A ***finite sample space*** $\Omega = \{\omega_1, ..., \omega_K\}$ with $K \in \mathbb{N}$.
    - Each ω represents a possible state of the economy/world. (mutually exclusive)
    - At $t = 0$ the investor does not know the state of the world.
    - Financial assets have a constant value at $t = 0$, but its value will depend on $\omega \in \Omega$ at time $t = 1$. (random variables)
3. A ***probability measure*** P (that is, a function $P : \Omega \to [0, 1]$ with $\sum_{i=1}^{K} P(\omega_i) = 1$), which we additionally assume to satisfy $P(\omega) > 0, \omega \in \Omega$.
4. A ***bank account process*** $B = \{B(t)\}_{t=0,1} = \{B(0), B(1)\}$, where with $B(0) = 1$ and $B(1)$ is a random variable with $B(1, \omega) > 0$. In fact, one usually finds that $B(1) \geq 1$.

Model specifications

Definition 1 (continuation)

Then, one has that

$$r = (B(1) - B(0)) / B(0) = B(1) - 1 \geq 0.$$

Moreover, a usual assumption is that $B(1)$ and r are constants.

5. A ***price process*** $S = \{S(t)\}_{t=0,1} = \{S(0), S(1)\}$ where

$$S(t) = (S_1(t), \cdots, S_N(t))^T,$$

and $N \geq 1$ is the number of risky assets.
You may think of these assets as stocks.
- At $t = 0$: the investor knows the value of the stocks, i.e., $S(0)$ are constants.
- At $t = 1$: the prices $S(1)$ are random variables, whose actual realizations become known to the investor only at time $t = 1$.

Model specifications

Definition 1 (continuation)

S represents the price of the risky assets because, usually, for all $j = 1, ..., N$ there exists $\omega_1(j)$ and $\omega_2(j)$ in Ω such that

$$S_j(1, \omega_1(j)) < S_j(0) < S_j(1, \omega_2(j)).$$

Note that $S_j(0) = S_j(0, \omega), \omega \in \Omega$, because $S_j(0)$ is constant.

Model specifications

Definition 2

A ***trading strategy*** is a vector $H = (H_0, H_1, \cdots, H_N)^T$, where

- $H_0 :=$ Amount of money invested in the bank account.
- $H_n :=$ Number of units of security n held between $t = 0$ and $t = 1$, $n = 1, \ldots, N$.

- Note that $H_n, n = 0, \ldots, N$ can be negative: borrowing/short selling.
- Moreover, $H_n, n = 0, \ldots, N$ are constants because these are decision taken at $t = 0$.

Model specifications

Definition 3

The ***value process*** $V = \{V(t)\}_{t=0,1}$, is the total value of the portfolio, associated to a trading strategy H, at each t, which is given by

$$V(t) = H_0 B(t) + \sum_{n=1}^{N} H_n S_n(t), \qquad t = 0, 1. \tag{1}$$

- Note that $V(0)$ is constant and $V(1)$ is a random variable.

Model specifications

Definition 4

The ***gain process*** G is the random variable describing the total profit/loss generated by a trading strategy H between $t = 0$ and $t = 1$ and is given by

$$\begin{aligned} G &= H_0\left(B\left(1\right) - B\left(0\right)\right) + \sum_{n=1}^{N} H_n\left(S_n\left(1\right) - S_n\left(0\right)\right) \\ &= H_0 r + \sum_{n=1}^{N} H_n \Delta S_n. \end{aligned} \tag{2}$$

- Note that
$$V\left(1\right) = V\left(0\right) + G. \tag{3}$$
- Moreover, the change in V is due to the changes in S, no addition/withdraw of funds allowed.

Model specifications

Definition 5

A ***numeraire*** is a financial asset used to measure the value of all other assets in the market, i.e., the price of all financial assets are expressed in units of numeraire.

- We will use the bank account as numeraire.
- As a consequence, $B(t) = 1, t = 0, 1$, and the quantities S, V and G will have their discounted versions (***normalized market***).

Definition 6

The discounted price process $S^* = \{S^*(t)\}_{t=0,1}$ is given by

$$S_n^*(t) = \frac{S_n(t)}{B(t)}, \qquad n = 1, ..., N, t = 0, 1. \tag{4}$$

Model specifications

Definition 7

The ***discounted value process*** $V^* = \{V^*(t)\}_{t=0,1}$ is given by

$$V^*(t) = \frac{V(t)}{B(t)}, \qquad n = 1, ..., N, t = 0, 1. \tag{5}$$

Definition 8

The ***discounted gains process*** G^* is given by

$$G^* = H_0(B^*(1) - B^*(0)) + \sum_{n=1}^{N} H_n(S_n^*(1) - S_n^*(0)) = \sum_{n=1}^{N} H_n \Delta S_n^*. \tag{6}$$

Moreover,

$$V^*(1) = V^*(0) + G^* \tag{7}$$

Model specifications

Definition 9

In a single period financial market model with $\#\Omega = K$ and N risky assets, the ***payoff matrix*** $S(1,\Omega)$ is defined to be

$$S(1,\Omega) = \begin{pmatrix} B(1,\omega_1) & S_1(1,\omega_1) & \cdots & S_N(1,\omega_1) \\ \vdots & \vdots & & \vdots \\ B(1,\omega_K) & S_1(1,\omega_K) & \cdots & S_N(1,\omega_K) \end{pmatrix} \in \mathbb{R}^{K\times(N+1)}.$$

- Note that, together with $B(0)$ and $S(0) = (S_1(0), ..., S_N(0))^T$, $S(1,\Omega)$ fully characterizes the market model.
- One can also consider the matrix

$$S(0,\Omega) = \begin{pmatrix} B(0) & S_1(0) & \cdots & S_N(0) \\ \vdots & \vdots & & \vdots \\ B(0) & S_1(0) & \cdots & S_N(0) \end{pmatrix} \in \mathbb{R}^{K\times(N+1)},$$

with the first row repeated K times.

Model specifications

- This way of specifying the market model emphasizes the linear algebra point of view on financial market models on finite probability spaces. That is:
    - Random variables are represented as elements in $\mathbb{R}^K$.
    - N random variables (or a N-dimensional random vector) are represented as elements in $\mathbb{R}^{K\times N}$.
    - Constants (degenerate random variables) can be represented as elements in $\mathbb{R}^K$ with all components being equal.
- We also consider the discounted payoff matrix $S^*(1,\Omega)$ in an obvious way.
- Note that $V(1), V^*(1), G, G^* \in \mathbb{R}^K$ associated to the trading strategy $H \in \mathbb{R}^{N+1}$ are given by

$$V(1) = S(1,\Omega)H, \qquad V^*(1) = S^*(1,\Omega)H,$$
$$G = \Delta S(\Omega)H, \quad \text{and} \quad G^* = \Delta S^*(\Omega)H,$$

where $\Delta S(\Omega) := S(1,\Omega) - S(0,\Omega)$, and $\Delta S^*(\Omega) := S^*(1,\Omega)$ $-S^*(0,\Omega)$.

Model specifications

- A probability measure Q can also be seen as an element in $\mathbb{R}^K$.
- Q induces a linear functional on the set of random variables $\mathbb{E}_Q[\cdot] : \mathbb{R}^K \to \mathbb{R}$, called expectation under Q, given by

$$\mathbb{E}_Q[Z] = \sum_{k=1}^{K} Q(\omega_k) Z(\omega_k) = \sum_{k=1}^{K} Q_k Z_k = Q^T Z = Z^T Q.$$

- The expected value of the random vector of (discounted) assets $\overline{S}(1) := (B(1), S_1(1), ..., S_N(1))^T$ is given by

$$\mathbb{E}_Q\left[\overline{S}(1)\right] = S^T(1,\Omega) Q, \qquad \left(\mathbb{E}_Q\left[\overline{S}^*(1)\right] = S^{*T}(1,\Omega) Q, \right).$$

- Note also that one can write the expected values of $V(1)$ and $V^*(1)$as

$$\mathbb{E}_Q[V(1)] = H^T S^T(1,\Omega) Q = Q^T S(1,H) H,$$
$$\mathbb{E}_Q[V^*(1)] = H^T S^{*T}(1,\Omega) Q = Q^T S^*(1,H) H.$$

Model specifications

Example 10

- Consider $N = 1, K = 2$ $(\Omega = \{\omega_1, \omega_2\})$, $r = 1/9$, $B(0) = 1$, $B(1) = 1 + r = \frac{10}{9}$, $S_1(0) = 5$ and

$$S_1(1,\omega) = \begin{cases} \frac{20}{3} & \text{if} \quad \omega = \omega_1 \\ \frac{40}{9} & \text{if} \quad \omega = \omega_2 \end{cases} = \frac{20}{3}\mathbf{1}_{\{\omega_1\}}(\omega) + \frac{40}{9}\mathbf{1}_{\{\omega_2\}}(\omega).$$

- The previous notation for $S_1(1)$ emphasizes the random variable nature of $S_1(1)$.
- You can also see $S_1(1)$ as an element of $\mathbb{R}^K = \mathbb{R}^2$, i.e., a column vector $S_1(1) = \left(\frac{20}{3}, \frac{40}{9}\right)^T$.
- The discounted price process is given by $S_1^*(0) = S_1(0)/B(0)$ $= 5/1 = 5$ and

$$S_1^*(1) = S_1(1)/B(1) = \left(\frac{\frac{20}{3}}{\frac{10}{9}}, \frac{\frac{40}{9}}{\frac{10}{9}}\right)^T = (6,4)^T.$$

Example 10

- Next consider a trading strategy $H = (H_0, H_1)^T$.
  - At $t = 0$: we have

$$\begin{aligned} V(0) &= H_0 B(0) + H_1 S_1(0) = H_0 + H_1 5, \\ V^*(0) &= H_0 + H_1 S_1^*(0) = H_0 + H_1 5. \end{aligned}$$

  - At $t = 1$: we have

$$\begin{aligned} V(1) &= H_0 B(1) + H_1 S_1(1) = \frac{10}{9} H_0 + H_1 S_1(1) \\ &= \begin{cases} \frac{10}{9} H_0 + \frac{20}{3} H_1 & \text{if} \quad \omega = \omega_1 \\ \frac{10}{9} H_0 + \frac{40}{9} H_1 & \text{if} \quad \omega = \omega_2 \end{cases}, \end{aligned}$$

$$\begin{aligned} V^*(1) &= H_0 + H_1 S_1^*(1) \\ &= \begin{cases} H_0 + 6 H_1 & \text{if} \quad \omega = \omega_1 \\ H_0 + 4 H_1 & \text{if} \quad \omega = \omega_2 \end{cases}, \end{aligned}$$

Model specifications

Example 10

$$G = H_0 r + H_1 \Delta S_1 = \frac{1}{9} H_0 + H_1 \left(S_1(1) - S_1(0)\right)$$
$$= \begin{cases} \frac{1}{9} H_0 + \left(\frac{20}{3} - 5\right) H_1 = \frac{1}{9} H_0 + \frac{5}{3} H_1 & \text{if} \quad \omega = \omega_1 \\ \frac{1}{9} H_0 + \left(\frac{40}{9} - 5\right) H_1 = \frac{1}{9} H_0 - \frac{5}{9} H_1 & \text{if} \quad \omega = \omega_2 \end{cases},$$

$$G^* = H_1 \Delta S_1^* = H_1 \left(S_1^*(1) - S_1^*(0)\right)$$
$$= \begin{cases} H_1 (6-5) = H_1 & \text{if} \quad \omega = \omega_1 \\ H_1 (4-5) = -H_1 & \text{if} \quad \omega = \omega_2 \end{cases},$$

- Please note that $V(1) = V(0) + G$ and $V^*(1) = V^*(0) + G^*$.

Arbitrage and Other Economic Considerations

Dominant trading strategies

- Financial markets are economically reasonable, which means that for sure profits do not exist.
- In real markets, those opportunities may exist for certain agents, but vanish quickly due to the action of arbitrageurs.
- This means that our financial market model must not allow for risk free profits.

Definition 11

A trading strategy $\widehat{H}$ is said to be a **dominant trading strategy** (**DTS**) if there exists another trading strategy $\widetilde{H}$ such that

$$\begin{cases} \widehat{V}(0) = \widetilde{V}(0) \\ \widehat{V}(1,\omega) > \widetilde{V}(1,\omega), \qquad \forall \omega \in \Omega \end{cases} \tag{8}$$

Dominant trading strategies

Lemma 12

The following statements are equivalent

1. $\exists$ ***DTS***.
2. $\exists$ *a trading strategy satisfying*

$$\begin{cases} V(0) = 0 \\ V(1,\omega) > 0, \qquad \forall \omega \in \Omega \end{cases}. \tag{9}$$

3. $\exists$ *a trading strategy satisfying*

$$\begin{cases} V(0) < 0 \\ V(1,\omega) \geq 0, \qquad \forall \omega \in \Omega \end{cases}. \tag{10}$$

Proof.

Smartboard. □

- If in 2. and/or 3. we change V by V^* the result still holds.

Dominant trading strategies

- The existence of a dominant trading strategy is also unsatisfactory because leads to “illogical” pricing.
- It is useful to interpret $V(1)$ as the payoff of a contingent claim (think of options) and $V(0)$ as the price of this claim.
- Assume that $\hat{H}$ dominates $\widetilde{H}$.
- Then, the prices $\widehat{V}(0)$ and $\widetilde{V}(0)$ coincide but the payoffs will satisfy

$$\widehat{V}(1,\omega) > \widetilde{V}(1,\omega), \qquad \omega \in \Omega.$$

- This clearly does not make sense as it provides a sure positive profit with zero initial investment by taking a long position in $\widehat{V}$ and a short position in $\widetilde{V}$.

Linear pricing measures

- The following concept is useful because it provides a "logical" pricing rule.

Definition 13

A ***linear pricing measure*** **(LPM)** is a non-negative vector $\pi = \left(\pi\left(\omega_1\right), ..., \pi\left(\omega_K\right)\right)^T$ such that for every trading strategy $H = \left(H_0, H_1, ..., H_N\right)^T$ the following holds

$$V^*\left(0\right) = \sum_{\omega \in \Omega} \pi\left(\omega\right) V^*\left(1, \omega\right). \tag{11}$$

- Note that equation (11) can be written as

$$H_0 + \sum_{n=1}^{N} H_n S_n^*\left(0\right) = \sum_{\omega \in \Omega} \pi\left(\omega\right) \left(H_0 + \sum_{n=1}^{N} H_n S_n^*\left(1, \omega\right)\right). \tag{12}$$

Linear pricing measures

Lemma 14

1. *Let* π *be a* **LPM**. *Then,* π *is a probability measure on* $\Omega = \{\omega_1, ..., \omega_K\}$.
2. π *is a* **LPM** $\Leftrightarrow$ π *is a probability measure satisfying*

$$S_n^* (0) = \sum_{\omega \in \Omega} S_n^* (1, \omega) \pi (\omega) =: \mathbb{E}_\pi [S_n^* (1)], \qquad n = 1, ..., N. \tag{13}$$

Proof.

Smartboard. □

Linear pricing measures

Remark 15

- The previous result says that

$$S_n^*(0) = \mathbb{E}_\pi [S_n^*(1)], \qquad n = 1, \ldots, N, \tag{14}$$

$$V^*(0) = \mathbb{E}_\pi [V^*(1)]. \tag{15}$$

- That is, the price/value at time 0 of a security can be obtained by taking expectations under a **LPM** π of the discounted terminal price/value of the security.
- In this context, equations (14) and (15) just say that the discounted processes S_n^* and V_n^* are martingales under π.
- Using a **LPM** each contingent claim $V(1, \omega)$ has a unique price and a claim that pays more than other for every $\omega \in \Omega$ will have a higher price (logical pricing).

Linear pricing measures and dominant trading strategies

Lemma 16

$\exists$ **LPM** $\Longleftrightarrow \nexists$ **DTS**.

Proof.

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- Financial market models allowing for **DTS** are not reasonable.
- But even less reasonable are models allowing for the failure of of the law of one price.

Law of one price

Definition 17

We say that the **law of one price** (**LOP**) holds for a financial market model if there do not exist two trading strategies $\widehat{H}$ and $\widetilde{H}$ such that

$$\begin{cases} \widehat{V}(0) > \widetilde{V}(0) \\ \widehat{V}(1,\omega) = \widetilde{V}(1,\omega), \qquad \forall \omega \in \Omega \end{cases} . \tag{16}$$

Remark 18

1. If in (16) we use $\widehat{V}^*$ and $\widetilde{V}^*$ we get the same concept.
2. **LOP** holds $\implies$ No ambiguity regarding the price at $t = 0$ ($V(0)$) of contingent claims ($V(1)$).
3. $\nexists$ two distinct trading strategies yielding the same payoff at $t = 1 \implies$ **LOP** holds.
4. **LOP** does not hold $\implies$ $\exists$ two distinct trading strategies with the same final value but different initial value.

Law of one price and dominant trading strategies

Lemma 19

$\nexists$ **DTS** $\Rightarrow$ **LOP** *holds.*

Proof.

- Suppose **LOP** does not hold. Then, there exist $\widehat{H}, \widetilde{H}$ such that $\widehat{V}^*(0) > \widetilde{V}^*(0)$ and $\widehat{V}^*(1) = \widetilde{V}^*(1)$.
- Since $\widehat{V}^*(1) = \widehat{V}^*(0) + \widehat{G}^*$ and $\widetilde{V}^*(1) = \widetilde{V}^*(0) + \widetilde{G}^*$, we have that $\widehat{G}^* < \widetilde{G}^*$.
- Define a new trading strategy H by setting $H_0 = -\sum_{n=1}^{N} H_n S_n^*(0)$, and $H_n = \widetilde{H}_n - \widehat{H}_n$, $n = 1, ..., N$.
- Then, $V^*(0) = H_0 + \sum_{n=1}^{N} H_n S_n^*(0) = 0$,

$$V^*(1) = V^*(0) + \sum_{n=1}^{N} \left(\widetilde{H}_n - \widehat{H}_n\right) \Delta S_n^* = \widetilde{G}^* - \widehat{G}^* > 0,$$

and by Lemma 12 there exists a **DTS**.

□

Law of one price and dominant trading strategies

Remark 20

1. **LOP** holds $\not\Longrightarrow$ $\nexists$ **DTS**. That is, the converse of the previous lemma does not hold. It is possible to have **DTS** and **LOP** still holds.
2. If in a model $\exists$ **DTS** the situation is bad because it leads to illogical pricing and the existence of strategies with a sure positive final value with zero initial investment.
3. If in a model **LOP** does not hold the situation is even worse. It also allows for the existence of "**suicide strategies**", that is, strategies with positive initial investment and sure zero final value. Let $\widehat{H}, \widetilde{H}$ such that $\widehat{V}(0) > \widetilde{V}(0)$ and $\widehat{V}(1) = \widetilde{V}(1)$. Then, by the linearity of V with respect to H, we have that $H := \widehat{H} - \widetilde{H}$ satisfies

$$V(0) = \widehat{V}(0) - \widetilde{V}(0) > 0 \qquad \text{and} \qquad V(1) = \widehat{V}(1) - \widetilde{V}(1) = 0.$$

Example LOP does not hold

Example 21

- Take $K = 2, N = 1, r = 1$, $B(0) = 1$, $B(1) = 2$, $S(0) = 10$ and

$$S(1, \omega) = \begin{cases} 12 & \text{if} \quad \omega = \omega_1 \\ 12 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

  That is, $S(1)$ is constant.

- Then,

$$\begin{aligned} V(0) &= H_0 B(0) + H_1 S(0) = H_0 + 10H_1, \\ V(1) &= H_0 B(1) + H_1 S(1) = 2H_0 + 12H_1. \end{aligned} \tag{17}$$

  Note that $V(1, \omega)$ is also constant.

- The previous linear system has a unique solution given by

$$H_0 = \frac{5}{4} V(1) - \frac{3}{2} V(0), \qquad H_1 = \frac{1}{4} V(0) - \frac{1}{8} V(1).$$

Example LOP does not hold

Example 21

- This means that, for fixed $V(1)$, there are an infinite number of strategies (each starting with a different $V(0)$) which yield $V(1)$ $\implies$ **LOP** does not hold.
- In the same model, suppose now that $S(1,\omega_2) = 8$.
- Now, in addition to (17) we have

$$\left.\begin{array}{c} V(1,\omega_1) = H_0 B(1) + H_1 S(1,\omega_1) = 2H_0 + 12H_1, \\ V(1,\omega_2) = H_0 B(1) + H_1 S(1,\omega_2) = 2H_0 + 8H_1. \end{array}\right\} \tag{18}$$

- For arbitrary $V(1,\omega_1)$ and $V(1,\omega_2)$ the system (18) has a unique solution and taking into account (17) we have that $V(0)$ is uniquely determined $\implies$ **LOP** holds.

Example LOP does not hold

Example 21

- However, for $H = (H_0, H_1)^T = (10, -1)^T$ we have

$$V(0) = H_0 + 10H_1 = 10 - 10 = 0,$$
$$V(1, \omega_1) = 2H_0 + 12H_1 = 20 - 12 = 8 > 0,$$
$$V(1, \omega_2) = 2H_0 + 12H_1 = 20 - 8 = 12 > 0.$$

- Hence, H is a **DTS**.

Arbitrage opportunity

Definition 22

An ***arbitrage opportunity*** (**AO**) is a trading strategy satisfying:

a) $V(0) = 0.$

b) $V(1,\omega) \geq 0, \quad \omega \in \Omega.$

c) $\mathbb{E}[V(1)] > 0.$

Remark 23

1. c) can be changed by

   c') $\exists\, \omega \in \Omega$ such that $V(1,\omega) > 0.$

2. a), b) c) $\Longleftrightarrow V^*(0) = 0, V^*(1) \geq 0$, and $\mathbb{E}[V^*(1)] > 0.$
3. An **AO** is a trading strategy
   - with zero initial investment,
   - without the possibility of bearing a loss
   - with a strictly positive profit for at least one of the possible states of the economy.

Arbitrage opportunity

Lemma 24

1. $\exists$ **DTS** $\implies$ $\exists$ **AO.**
2. $\exists$ **AO** $\not\Longrightarrow$ $\exists$ **DTS.**

Proof.

1. By Lemma 12, we know that

   $\exists$ of **DTS** $\iff$ $\exists$ of H such that $V(0) = 0$ and $V(1, \omega) > 0, \omega \in \Omega$.

   But, if $V(1, \omega) > 0, \omega \in \Omega$ then

   $$\mathbb{E}[V(1)] = \sum_{\omega \in \Omega} V(1, \omega) P(\omega) > 0.$$

2. The following example provides a counterexample.

□

Arbitrage opportunity

Example 25

- Take $K = 2, N = 1, r = 0, B(0) = 1, B(1) = 1, S(0) = S^*(0) = 10$ and
$$S(1,\omega) = S^*(1,\omega) = \begin{cases} 12 & \text{if} \quad \omega = \omega_1 \\ 10 & \text{if} \quad \omega = \omega_2 \end{cases}.$$
- Consider the trading strategy $H = (H_0, H_1)^T = (-10, 1)^T$, then $V(0) = H_0 B(0) + H_1 S(0) = -10 + 10 = 0$, and
$$V(1) = H_0 B(1) + H_1 S(1) = \begin{cases} -10 + 12 = 2 & \text{if} \quad \omega = \omega_1 \\ -10 + 10 = 0 & \text{if} \quad \omega = \omega_2 \end{cases}.$$
- Hence, H is an arbitrage opportunity.

Example 25

- By Lemma 16 we know that the model does not contain **DTS** if and only if $\exists$ **LPM**.
- A **LPM** $\pi = (\pi_1, \pi_2)^T$ must satisfy $\pi \geq 0$ and

$$10 = S^* (0) = \mathbb{E}_\pi [S^* (1)] = 12\pi_1 + 10\pi_2.$$

- Hence, $\pi = (0, 1)^T$ is a **LPM** and we can conclude.

Arbitrage opportunity

Lemma 26

H *is an* ***AO*** $\iff G^*(\omega) \geq 0, \omega \in \Omega$ *and* $\mathbb{E}[G^*] > 0.$

Proof.

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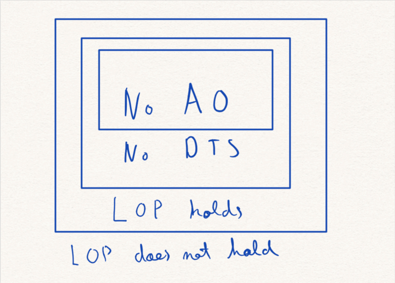
Remark 27

All single period securities market model can be classified in four categories



Risk Neutral Probability Measures

Risk neutral probability measures

- Recall that $\exists$ **LPM** $\implies \nexists$ **DTS**, but there may be **AO**.
- In order to rule out **AO** we need to narrow the concept of **LPM**.
- The idea is to require that a **LPM** must assign a strictly positive probability to each state of the economy.
- Equivalently, a **LPM**, say π, must be equivalent to P, that is,

$$P(\omega) > 0 \iff \pi(\omega) > 0, \qquad \omega \in \Omega.$$

Definition 28

A probability measure Q is called a ***risk neutral probability measure*** (**RNPM**) if

1. $Q(\omega) > 0, \qquad \omega \in \Omega.$
2. $\mathbb{E}_Q[\Delta S_n^*] = 0, \qquad n = 1, ..., N.$

Given a financial market model, we will denote by $\mathbb{M}$ the set of all **RNPM**.

Risk neutral probability measures

Remark 29

- Observe that

$$0 = \mathbb{E}_Q\left[\Delta S_n^*\right] = \mathbb{E}_Q\left[S_n^*(1) - S_n^*(0)\right] = \mathbb{E}_Q\left[S_n^*(1)\right] - S_n^*(0).$$

- That is, $\mathbb{E}_Q\left[S_n^*(1)\right] = S_n^*(0)$.
- Therefore, Q is a **LPM**.

Theorem 30 (First Fundamental Theorem of Asset Pricing (FFTAP))

$\nexists$ **AO** $\Longleftrightarrow$ $\exists$ **RNPM** *(that is,* $\mathbb{M} \neq \emptyset$*).*

Proof.

Smartboard. □

Risk neutral probability measures

Example 31 ($\exists!$ RNPM)

- Take $K = 2, N = 1, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S(0) = 5$, and

$$S^*(1,\omega) = \begin{cases} 6 & \text{if} \quad \omega = \omega_1 \\ 4 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- We are seeking a probability measure $Q = (Q_1, Q_2)^T$ such that

$$\mathbb{E}_Q[\Delta S^*] = 0 \Longleftrightarrow \mathbb{E}_Q[S^*(1)] = S^*(0) = 5$$
$$\Longleftrightarrow \begin{cases} 6Q_1 + 4Q_2 & = 5 \\ Q_1 + Q_2 & = 1 \end{cases}.$$

- $\exists!$ solution to the previous equation given by $Q = (1/2, 1/2)$.
- Therefore, Q is a **RNPM** and the market is arbitrage free by the **FFTAP**.

Risk neutral probability measures

Example 32 ($\exists\infty$ RNPM)

- Take $K = 3, N = 1, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S(0) = 5$, and

$$S^*(1,\omega) = \begin{cases} 6 & \text{if} \quad \omega = \omega_1 \\ 4 & \text{if} \quad \omega = \omega_2 \\ 3 & \text{if} \quad \omega = \omega_3 \end{cases}.$$

- For $Q = (Q_1, Q_3, Q_3)^T$ to be a **RNPM**, Q must satisfy

$$\mathbb{E}_Q[\Delta S^*] = 0 \iff \mathbb{E}_Q[S^*(1)] = S^*(0) = 5$$
$$\iff \begin{cases} 6Q_1 + 4Q_2 + 3Q_3 & = 5 \\ Q_1 + Q_2 + Q_3 & = 1 \end{cases}.$$

- We have 2 equations and 3 unknowns (underdetermined system).

Example 32 ($\exists\infty$ RNPM)

- In addition, we also have the restrictions $Q_i > 0, i = 1, 2, 3$.
- Solving the equations, taking into account the constraints, we obtain a family of **RNPM**

$$Q_\lambda = (\lambda, 2 - 3\lambda, -1 + 2\lambda)^T, \qquad \lambda \in (1/2, 2/3).$$

- Now there are infinitely many **RNPM** (one for each λ) and, again, the market is arbitrage free by the **FFTAP**.

Risk neutral probability measures

Example 33

- Take
  $K = 3, N = 2, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S_1(0) = 5, S_2(0) = 10,$

$$S_1^*(1,\omega) = \begin{cases} 6 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3 \end{cases},$$

  and

$$S_2^*(1,\omega) = \begin{cases} 12 & \text{if} \quad \omega = \omega_1 \\ 8 & \text{if} \quad \omega = \omega_2 \\ 8 & \text{if} \quad \omega = \omega_3 \end{cases}.$$

- We study this market model on the smartboard.

Valuation of Contingent Claims

Valuation of contingent claims

Definition 34

A ***contingent claim*** is a random variable X representing a payoff at time $t = 1$.

- Think of a contingent claim as any financial contract with some payoff at time $t = 1$ (options for instance).

Definition 35

A contingent claim is said to be ***attainable*** (or ***marketable***) if there exists a trading strategy H, called the ***replicating/hedging*** portfolio, such that $V(1) = X$. We say that H ***generates/replicates/hedge*** X.

Valuation of contingent claims

- Suppose that the contingent claim X is attainable, i.e., $V(1) = X$.
- Suppose also that it can be bought in the market (at time 0) for the price $p(X)$.
- Then, using the no arbitrage pricing principle:
  - If $p(X) > V(0)$:
    - At $t = 0$: Sell the claim (receive $p(X)$), implement X (that is, $V(1)$ at cost $V(0)$) and invest $p(X) - V(0)$ risk free.
    - At $t = 1 : -X + V(1) + (p(X) - V(0))(1 + r) > 0$.
  - If $p(X) < V(0)$:
    - At $t = 0$: Buy the claim (pay $p(X)$), implement $-X$ (that is, $-V(1)$ receiving $V(0)$) and invest $V(0) - p(X)$ risk free.
    - At $t = 1 : X - V(1) + (V(0) - p(X))(1 + r) > 0$.
- Does this mean that $p(X) = V(0)$ is the correct price for X? Not necessarily.
- Suppose that $\exists \widehat{H}$ such that $\widehat{V}(1) = X$ and $\widehat{V}(0) \neq V(0)$.
- This second strategy could be used to generate an arbitrage if $p(X) = V(0)$.

Valuation of contingent claims

- In order to rule out this possibility we need to assume that **LOP** holds.
- We have just proved the following result.

Proposition 36

If **LOP** *holds, then the price $p(X)$ ($t = 0$ value) of an attainable contingent claim X is given by*

$$p(X) = V(0) = H_0 B(0) + \sum_{n=1}^{N} H_n S_n(0), \tag{19}$$

where H is any trading strategy that generates X.

- Recall that $\nexists$ **AO** $\implies$ $\nexists$ **DTS** $\implies$ **LOP** holds.
- By the **FFTAP,** we also have that if $\mathbb{M} \neq \emptyset$ then $\nexists$ **AO** (and **LOP** holds).

Valuation of contingent claims

Theorem 37

Assume $\nexists$ **AO***. Then, the price* $p(X)$ *of any attainable contingent claim* X *is given by*

$$p(X) = \mathbb{E}_Q\left[\frac{X}{B(1)}\right], \tag{20}$$

where Q *is any* **RNPM** *in* $\mathbb{M}$ *.*

Proof.

Smartboard. □

Example 38 (Continuation Example 31)

- Take $K = 2, N = 1, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S(0) = 5,$

$$S^*(1,\omega) = \begin{cases} 6 & \text{if} \quad \omega = \omega_1 \\ 4 & \text{if} \quad \omega = \omega_2 \end{cases},$$

  and

$$S(1,\omega) = \begin{cases} 6\frac{10}{9} = \frac{20}{3} & \text{if} \quad \omega = \omega_1 \\ 4\frac{10}{9} = \frac{40}{9} & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- Recall that in this market there is only one RNPM $Q = (1/2, 1/2)^T$.
- Let X be the contingent claim defined by

$$X(\omega) = \begin{cases} 7 & \text{if} \quad \omega = \omega_1 \\ 2 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

Example 38

- Suppose that X is attainable, then the price of X is given by

$$p(X) = \mathbb{E}_Q\left[\frac{X}{B(1)}\right] = \frac{7}{\frac{10}{9}}\frac{1}{2} + \frac{2}{\frac{10}{9}}\frac{1}{2} = \frac{81}{20}.$$

- Let's prove that X is indeed attainable. We want to find $H = (H_0, H_1)^T$ that generates X, that is,

$$\frac{X}{B(1)} = V^*(1) = V^*(0) + G^* = V^*(0) + H_1 \Delta S^*.$$

- Since $V^*(0) = V(0) = p(X) = \frac{81}{20}$ and

$$\Delta S^* = \begin{cases} 6 - 5 = 1 & \text{if} \quad \omega = \omega_1 \\ 4 - 5 = -1 & \text{if} \quad \omega = \omega_2 \end{cases},$$

Example 38

we get the following equations

$$\frac{7}{\frac{10}{9}} = \frac{81}{20} + H_1,$$
$$\frac{2}{\frac{10}{9}} = \frac{81}{20} - H_1.$$

- These two equations are compatible and $H_1 = \frac{9}{4}$.
- To determine H_0 we can use

$$\frac{81}{20} = V(0) = H_0 B(0) + H_1 S(0) = H_0 + \frac{9}{4}5,$$

which yields $H_0 = -\frac{36}{5}$.

Example 38

- The interpretation is as follows:
  - At $t = 0$:
    - You sell the claim and get $V(0) = \frac{81}{20}$.
    - You hedge the claim by borrowing $-H_0 = \frac{36}{5}$ at interest $\frac{1}{9}$, using $V(0) - H_0 = \frac{81}{20} + \frac{36}{5} = \frac{45}{4}$ to buy $H_1 = \frac{V(0)-H_0}{S(0)} = \frac{\frac{45}{4}}{5} = \frac{9}{4}$ shares of the stock.
  - At $t = 1$:
    - Pay $-H_0 B(1) = \frac{36}{5}\frac{10}{9} = 8$ to the bank to close the loan.
    - The value of the portfolio is

$$V(1) = H_0 B(1) + H_1 S(1) = -8 + \frac{9}{4} S(1)$$
$$= \begin{cases} -8 + \frac{9}{4}\frac{20}{3} = 7 & \text{if} \quad \omega = \omega_1 \\ -8 + \frac{9}{4}\frac{40}{9} = 2 & \text{if} \quad \omega = \omega_2 \end{cases},$$

and you can pay the contingent claim sold.

Example 38

- Now, suppose that we add a third state ω_3 in the economy and $S^*(1, \omega_3) = 3$ and $S(1, \omega_3) = \frac{10}{3}$.
- This is the same extension as in Example 32, so we know $\exists\infty$ **RNPM.**
- Consider an arbitrary contingent claim X in this market, that is,

$$X(\omega) = \begin{cases} X_1 & \text{if} \quad \omega = \omega_1 \\ X_2 & \text{if} \quad \omega = \omega_2 \\ X_3 & \text{if} \quad \omega = \omega_3 \end{cases} = (X_1, X_2, X_3)^T.$$

- X is attainable if there exists $H = (H_0, H_1)^T$ such that

$$X = V(1) = H_0 B(0) + H_1 S(1).$$

Valuation of contingent claims

Example 38

- The previous vector equation boils down to the following overdetermined linear system

$$\begin{cases} X_1 &= \frac{10}{9}H_0 + \frac{20}{3}H_1 \\ X_2 &= \frac{10}{9}H_0 + \frac{40}{9}H_1 \\ X_3 &= \frac{10}{9}H_0 + \frac{10}{3}H_1 \end{cases} .$$

- From the first equation we obtain $\frac{10}{9}H_0 = X_1 - \frac{20}{3}H_1$ and substituting this expression for $\frac{10}{9}H_0$ in the second and third equations we get

$$\begin{cases} X_2 &= X_1 - \frac{20}{3}H_1 + \frac{40}{9}H_1 = X_1 - \frac{20}{9}H_1 \\ X_3 &= X_1 - \frac{20}{3}H_1 + \frac{10}{3}H_1 = X_1 - \frac{10}{3}H_1 \end{cases} .$$

Valuation of contingent claims

Example 38

- The first equation in the previous system gives

$$H_1 = \frac{9}{20}\left(X_2 - X_1\right),$$

and the second equation gives

$$H_1 = \frac{3}{10}\left(X_3 - X_1\right).$$

- Therefore, equating the previous expressions for H_1, we obtain.

$$\frac{9}{20}\left(X_2 - X_1\right) = \frac{3}{10}\left(X_3 - X_1\right) \iff X_1 - 3X_2 + 2X_3 = 0. \qquad (21)$$

- We can conclude that a contingent claim $X = (X_1, X_2, X_3)^T$ in this market is attainable if and only if X satisfies equation (21).

Valuation of contingent claims

Example 39

- In a general single period model consider the so called ***counting claim*** X defined by

$$X(\omega) = \begin{cases} 1 & \text{if} \quad \omega = \widehat{\omega} \\ 0 & \text{if} \quad \omega \neq \widehat{\omega} \end{cases},$$

  for some $\widehat{\omega} \in \Omega$.
- Assuming that X is attainable we have that

$$p(X) = \mathbb{E}_Q\left[\frac{X}{B(1)}\right] = \sum_{\omega \in \Omega} \frac{X(\omega)}{B(1)} Q(\omega) = \frac{Q(\widehat{\omega})}{B(1)} =: p(\widehat{\omega}).$$

- $p(\widehat{\omega})$ is called the state price for state $\widehat{\omega}$.
- The price of any contingent claim X can be obtained as the weighted sum of its payoff where the weights are the state prices, i.e., $p(X) = \sum_{\omega \in \Omega} X(\omega)\, p(\omega)$.

Complete and Incomplete Markets

Definition 40

A financial market model is ***complete*** if every contingent claim X is attainable.

Otherwise, we say that the market model is ***incomplete***.

- So far, in order to use the risk neutral pricing principle to find the price of a contingent claim X, we need to ensure that the contingent claim is attainable.
- Therefore, it is important to find useful criteria to decide if a claim is attainable and, more generally, if the market is complete.
- Recall that $S(1, \Omega)$ is the payoff matrix introduced in Definition 9 and $K = \#\Omega$.

Lemma 41

The market is complete $\Longleftrightarrow \operatorname{rank}(S(1,\Omega)) = K$.

Proof.

- Let $H = (H_0, H_1, ..., H_n)^T \in \mathbb{R}^{N+1}$ be a trading strategy and $X = (X_1, ..., X_K)^T \in \mathbb{R}^K$ a contingent claim.
- The market is complete $\Longleftrightarrow S(1,\Omega)\,H = X$ has a solution in H for every $X \Longleftrightarrow$ Linear span of the columns of $S(1,\Omega)$ is $\mathbb{R}^K \Longleftrightarrow \dim\left(\operatorname{col}\left(S\left(1,\Omega\right)\right)\right) = K$.
- But note that

$$\operatorname{rank}(S(1,\Omega)) = \dim\left(\operatorname{col}\left(S\left(1,\Omega\right)\right)\right) = \dim\left(\operatorname{row}\left(S\left(1,\Omega\right)\right)\right).$$

- That is, if $S(1,\Omega)$ has K linear independent columns or rows.

$\square$

Example 42 (Continuation of Example 31)

- Take $K = 2, N = 1, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S_1(0) = 5$, and

$$S_1(1, \omega) = \begin{cases} \frac{20}{3} & \text{if} \quad \omega = \omega_1 \\ \frac{40}{9} & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- Recall that this market is arbitrage free and it has a unique **RNPM** given by $Q = \left(\frac{1}{2}, \frac{1}{2}\right)^T$.
- Moreover,

$$S(1, \Omega) = \begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ \frac{10}{9} & \frac{40}{9} \end{pmatrix} \sim_{R_2 \rightsquigarrow R_2 - R_1} \begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ 0 & \frac{-20}{9} \end{pmatrix},$$

and we can conclude that $\operatorname{rank}(S(1, \Omega)) = 2 = K$ and the market is complete.

Example 42

- In the same market we add a second asset with $S_2(0) = 54$ and

$$S_2(1,\omega) = \begin{cases} 70 & \text{if} \quad \omega = \omega_1 \\ 50 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- We have that

$$\mathbb{E}_Q\left[S_2^*(1)\right] = \frac{70}{\frac{10}{9}}\frac{1}{2} + \frac{50}{\frac{10}{9}}\frac{1}{2} = 54 = S_2^*(0),$$

and, therefore, Q is also a **RNPM** in the extended market.

- Moreover,

$$S(1,\Omega) = \begin{pmatrix} \frac{10}{9} & \frac{20}{3} & 70 \\ \frac{10}{9} & \frac{40}{9} & 50 \end{pmatrix} \sim_{R_2 \rightsquigarrow R_2 - R_1} \begin{pmatrix} \frac{10}{9} & \frac{20}{3} & 70 \\ 0 & \frac{-20}{9} & -20 \end{pmatrix},$$

so the $\operatorname{rank}(S(1,\Omega)) = \dim(\operatorname{row}(S(1,\Omega))) = 2 = K$ and the market is also complete.

Example 43 (Continuation of Example 32)

- Take $K = 3, N = 1, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S(0) = 5$, and

$$S^*(1,\omega) = \begin{cases} 6 & \text{if} \quad \omega = \omega_1 \\ 4 & \text{if} \quad \omega = \omega_2 \\ 3 & \text{if} \quad \omega = \omega_3 \end{cases}.$$

- In this market we have a family of **RNPM**

$$Q_\lambda = (\lambda, 2 - 3\lambda, 2\lambda - 1)^T, \qquad \lambda \in (1/2, 2/3).$$

- Moreover, the market is incomplete since

$$S(1,\Omega) = \begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ \frac{10}{9} & \frac{40}{9} \\ \frac{10}{9} & \frac{30}{9} \end{pmatrix} \sim^{R_2 \rightsquigarrow R_2 - R_1}_{R_3 \rightsquigarrow R_3 - R_1} \begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ 0 & -\frac{20}{9} \\ 0 & -\frac{30}{9} \end{pmatrix},$$

and the $\operatorname{rank}(S(1,\Omega)) = \dim(\operatorname{col}(S(1,\Omega))) = 2 \neq K = 3$.

Example 43

- For any contingent claim X and any **RNPM** Q_λ we have

$$\begin{aligned}\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] &= \lambda\frac{9}{10}X_1 + (2-3\lambda)\frac{9}{10}X_2 + (2\lambda-1)\frac{9}{10}X_3\\ &= \frac{9}{10}\lambda(X_1 - 3X_2 + 2X_3) + \frac{9}{10}(2X_2 - X_1).\end{aligned}$$

- If X is attainable this value must be the same for all $\lambda \in \left(\frac{1}{2}, \frac{2}{3}\right)$ because it must coincide with $V(0)$, which does not depend on Q_λ.
- Note that this happens if and only if

$$X_1 - 3X_2 - 2X_3 = 0.$$

- Recall (see Example 38) that this condition also characterizes the attainable contingent claims in this market.
- This is a general principle.

Complete and Incomplete Markets

Lemma 44

Suppose that $\mathbb{M} \neq \emptyset$. *Then,*

A contingent claim X *is attainable* $\iff \mathbb{E}_Q\left[\frac{X}{B(1)}\right]$ *is constant with respect to* $Q \in \mathbb{M}$.

Proof.

Smartboard. □

Theorem 45 (Second Fundamental Theorem of Asset Pricing (SFTAP))

Suppose that $\mathbb{M} \neq \emptyset$. *Then,*

The market model is complete $\iff \mathbb{M} = \{Q\}$, *that is,* $\exists!$ **RNPM**.

Proof.

Smartboard. □

Complete and Incomplete Markets

- Summarizing, we know how to price all attainable claims in a single period financial market.
- But, what about non-attainable claims in an incomplete model?
- We need some new concepts.

Definition 46

Let X be a non-attainable contingent claim. Then,

1. The **upper hedging price** of X, denoted by $V_+(X)$, is defined as

$$V_+(X) := \inf\left\{\mathbb{E}_Q\left[\frac{Y}{B(1)}\right] : Y \geq X, \quad Y \text{ is attainable}\right\}.$$

2. The **lower hedging price** of X, denoted by $V_-(X)$, is defined as

$$V_-(X) := \sup\left\{\mathbb{E}_Q\left[\frac{Y}{B(1)}\right] : Y \leq X, \quad Y \text{ is attainable}\right\}.$$

Remark 47 (An analogous remark apply to $V_{-}(X)$)

1. $V_{+}(X)$ is well defined and it is finite.
   - For any $\lambda > 0$, $\lambda B(1)$ is an attainable claim and if λ is large enough ($\lambda = \max_k \left\{\frac{X_k}{B(1)}\right\}$) we have $\lambda B(1) \geq X$.
   - Hence, $V_{+}(X) \leq \mathbb{E}_Q\left[\frac{\lambda B(1)}{B(1)}\right] = \lambda < +\infty$.
   - We also have that

$$\begin{aligned} V_{+}(X) &:= \inf_{Y \geq X, Y \text{ is attainable}} \left\{\mathbb{E}_Q\left[\frac{Y}{B(1)}\right]\right\} \\ &\geq \inf_{Y \geq X, Y \text{ is attainable}} \left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right]\right\} \\ &= \mathbb{E}_Q\left[\frac{X}{B(1)}\right] \geq \min_k \left\{\frac{X_k}{B(1)}\right\} > -\infty. \end{aligned}$$

   - Since this inequality holds for all $Q \in \mathbb{M}$, it follows that

$$V_{+}(X) \geq \sup\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right] : Q \in \mathbb{M}\right\}.$$

Remark

2
- $V_+(X)$ provides a good upper bound on the fair price of X in the sense that is the price of the cheapest portfolio that can be used to hedge a short position on X.
- If you sell the contingent claim X for more than $V_+(X)$ you can make a risk-less profit.

- Therefore, the fair price of X must lie in the interval $[V_-(X), V_+(X)]$.
- So we are interested in computing $V_+(X)$ as well as any attainable contingent claim $Y \geq X$ such that $V_+(X) = \mathbb{E}_Q\left[\frac{Y}{B(1)}\right]$.

Theorem 48

If $\mathbb{M} \neq \emptyset$, then for any contingent claim X one has

$$V_{+}(X) = \sup\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right] : Q \in \mathbb{M}\right\}$$

and

$$V_{-}(X) = \inf\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right] : Q \in \mathbb{M}\right\}.$$

Note that if X is attainable

$$V_{+}(X) = V_{-}(X) = \mathbb{E}_Q\left[\frac{X}{B(1)}\right],$$

for any $Q \in \mathbb{M}$.

Example 49 (Continuation Examples 32 and 43)

- Consider the market with $B(0) = 1, S(0) = 5$ and payoff matrix

$$S(1,\Omega) = \begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ \frac{10}{9} & \frac{40}{9} \\ \frac{10}{9} & \frac{30}{9} \end{pmatrix}.$$

- In this market we have a family of **RNPM**

$$\mathbb{M} = \left\{ Q_\lambda = (\lambda, 2-3\lambda, 2\lambda-1)^T,\, \lambda \in \left(\frac{1}{2}, \frac{2}{3}\right) \right\},$$

  and $X = (X_1, X_2, X_3)^T$ is attainable if and only if

$$X_1 - 3X_2 - 2X_3 = 0.$$

- Take $X = (30, 20, 10)^T$, which is not attainable because $30 - 3 \times 20 - 2 \times 10 \neq -50$.

Example 49

- Then, we compute

$$\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] = \lambda \frac{9}{10}30 + (2-3\lambda)\frac{9}{10}20 + (2\lambda-1)\frac{9}{10}10$$
$$= 27 - 9\lambda.$$

- This gives

$$V_+(X) = \sup_{Q\in\mathbb{M}}\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right]\right\} = \sup_{\lambda\in\left(\frac{1}{2},\frac{2}{3}\right)}\{27-9\lambda\}$$
$$= 27 - 9\frac{1}{2} = 22.5,$$
$$V_-(X) = \inf_{Q\in\mathbb{M}}\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right]\right\} = \inf_{\lambda\in\left(\frac{1}{2},\frac{2}{3}\right)}\{27-9\lambda\}$$
$$= 27 - 9\frac{2}{3} = 21.$$

Example 49

- Any price of X in the interval $[21, 22.5]$ is arbitrage free.
- By solving appropriate **LP** problems one can find attainable claims corresponding to the upper and lower hedging prices $V_+(X)$ and $V_-(X)$.
- In fact, one can check that
  - $Y = (30, 20, 15)^T \geq (30, 20, 10)^T = X$ gives
$$V_+(X) = \mathbb{E}_{Q_\lambda}\left[\frac{Y}{B(1)}\right], \qquad \lambda \in \left(\frac{1}{2}, \frac{2}{3}\right).$$
  - $Y = \left(30, \frac{50}{3}, 10\right)^T \leq (30, 20, 10)^T = X$ gives
$$V_-(X) = \mathbb{E}_{Q_\lambda}\left[\frac{Y}{B(1)}\right], \qquad \lambda \in \left(\frac{1}{2}, \frac{2}{3}\right).$$

Optimal Portfolio Problem (OPP)

Introduction

- The goal of an investor is transforming wealth invested at time $t = 0$ into wealth at time $t = 1$.
- The goal in this section will be to choose the "best" trading strategy.
- To be able to talk about "best" we need a measure of performance.
- We need to introduce the concept of utility function.

Utility functions

Definition 50 (Utility function)

A functions $U : \mathbb{R} \times \Omega \to \mathbb{R}$ is called a **utility function** if for each $\omega \in \Omega$ fixed the function $u \mapsto U(u, \omega)$ is

1. differentiable,
2. concave,
3. strictly increasing $\left(\frac{\partial}{\partial u} U(u, \omega) > 0, \omega \in \Omega\right)$.

- For many applications it suffices for U to depend only on the wealth argument u and not on $\omega \in \Omega$.

Utility functions

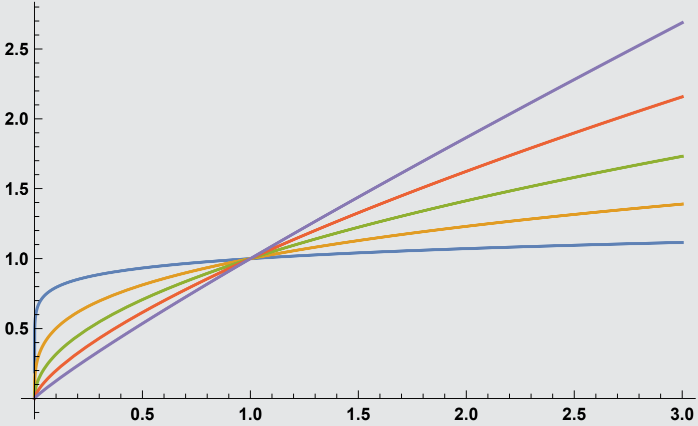
Remark 51

- If $V(1)$ is the portfolio value at $t = 1$, then $U(V(1))$ represents the utility of the wealth $V(1)$. $(U(V(1,\omega),\omega), \omega \in \Omega)$.
- U being increasing: More wealth $\Longrightarrow$ More utility.
- U being concave: More wealth $\Longrightarrow$ Less marginal utility (saturation effect)
- Our measure of performance will be the expected utility of the final wealth, that is,

$$\mathbb{E}[U(V(1))] = \sum_{\omega \in \Omega} U(V(1,\omega),\omega) P(\omega).$$

Example 52 (Utility functions)

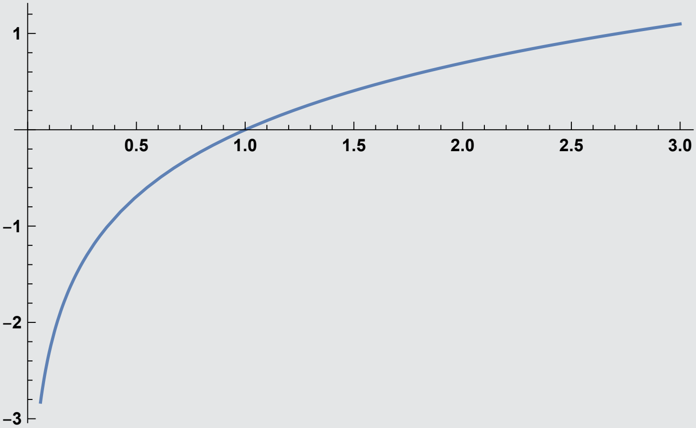
- $U_1(u) = u^\gamma, \quad u > 0, \gamma \in (0,1)$.



Utility functions

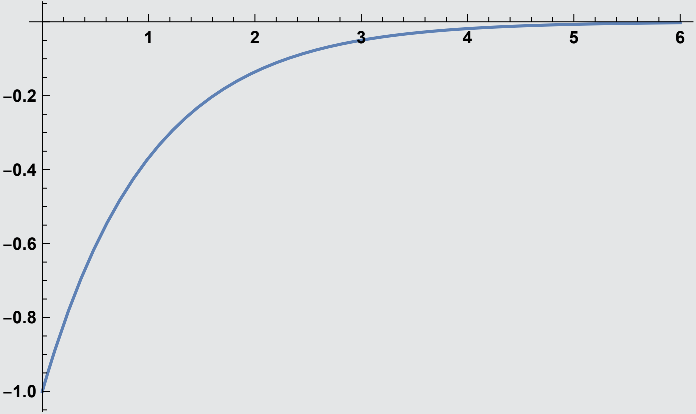
Example 52

- $U_2(u) = \log(u), \quad u > 0.$



Example 52

- $U_3(u) = -e^{-u}, \quad u > 0.$



Optimization problem

- Given an initial wealth $v \in \mathbb{R}$, we can consider the set of strategies $H \in \mathbb{R}^{N+1}$ such that

$$v = H_0 B(0) + \sum_{n=1}^{N} H_n S_n(0),$$

which impose some constraints on H, and try to maximize the expected utility of the terminal wealth.

- That is,

Optimal Portfolio Problem $(\mathrm{OPP}(v, U))$

$$\left.\begin{array}{cc} \max & \mathbb{E}\left[U\left(V(1)\right)\right] \\ \text{subject to} & V(0) = v \in \mathbb{R}, \\ & H \in \mathbb{R}^{N+1}, \end{array}\right\} \qquad (22)$$

Optimization problem

- Taking into account that $V(1) = B(1)\,V^*(1)$ and

$$V^*(1) = V^*(0) + G^* = v + \sum_{n=1}^{N} H_n \Delta S_n^*,$$

we can transform the previous optimization problem with contraints to an unconstrained one.

- That is,

Unconstrained Optimal Portfolio Problem $(\mathrm{UOPP}(v, U))$

$$\max_{(H_1,\ldots,H_N)^T \in \mathbb{R}^N} \mathbb{E}\left[U\left(B(1)\left\{v + \sum_{n=1}^{N} H_n \Delta S_n^*\right\}\right)\right] \quad (23)$$

- Note that we just have moved the inital wealth v from the constrain to the functional to optimize, eliminating the constraint and reducing the arguments of the functional by one.

Optimal portfolio problem and arbitrage opportunities

- Given a solution to **UOPP**(v, U) we get a solution to **OPP**(v, U) using $v = H_0 B(0) + \sum_{n=1}^{N} H_n S_n(0)$, and viceversa.

Lemma 53

$\exists$ *solution to the* **OPP**$(v, U) \implies \nexists$ **AO**.

Proof.

Smartboard. □

Remark 54

The previous result also tells us that if $\exists$ an optimal solution to the portfolio problem then $\mathbb{M} \neq \emptyset$.

Optimal portfolio problem and RNPM

Lemma 55

Suppose H is a solution to the **OPP**(v, U) *and $V(1)$ is its final value. Then,*

$$Q(\omega) = \frac{B(1,\omega)\, U'(V(1,\omega),\omega)}{\mathbb{E}[B(1)\, U'(V(1))]} P(\omega), \quad \omega \in \Omega,$$

is a **RNPM**.

Proof.

Smartboard. □

Definition 56

Let $Q \in \mathbb{M}$, then $L = Q/P$ is called the **state price density/ vector** (associated to Q).

Remark 57

Suppose $B(1) = B(0)(1+r)$ is constant, H is a solution to the **OPP**(v, U) and $V(1)$ is its final value. Then,

$$L(\omega) = \frac{Q(\omega)}{P(\omega)} = \frac{U'(V(1,\omega),\omega)}{\mathbb{E}[U'(V(1))]}, \quad \omega \in \Omega,$$

that is, the state price density is proportional to the marginal utility of the terminal wealth ($U'(V(1))$).

Viability of a market

- What about the converse of Lemma 55?
- If there exists a **RNPM** Q, then does the **OPP**(v, U) have a solution?
- Not necessarily, for some v and U it may happen that **OPP**(v, U) does not have a solution.
- However, one can always find a pair (v, U) such that **OPP**(v, U) has a solution.

Viability of a market

Definition 58

A market model is **viable** if there exists a function $U : \mathbb{R} \times \Omega \to \mathbb{R}$ and an initial wealth v such that $u \mapsto U(u, \omega)$ is concave, strictly increasing and differentiable for each $\omega \in \Omega$ and such that the corresponding **OPP**(v, U) has a solution.

Proposition 59

A market model is viable $\iff \mathbb{M} \neq \emptyset$.

Proof.

Smartboard. □

Example of OPP

Example 60

- Take a generic market model with $N = 2$ and $K = 3$.
- Consider the utility function $U(u) = -e^{-u}$,with derivative $U'(u) = e^{-u}$.
- Then, at a maximum the following equation must hold

$$\begin{aligned}
0 &= \frac{\partial}{\partial H_1}\mathbb{E}\left[U\left(B(1)\left\{v + H_1\Delta S_1^* + H_2\Delta S_2^*\right\}\right)\right] \\
&= \mathbb{E}\left[\Delta S_1^* \exp\left(-B(1)\left\{v + H_1\Delta S_1^* + H_2\Delta S_2^*\right\}\right)\right], \\
0 &= \frac{\partial}{\partial H_2}\mathbb{E}\left[U\left(B(1)\left\{v + H_1\Delta S_1^* + H_2\Delta S_2^*\right\}\right)\right] \\
&= \mathbb{E}\left[\Delta S_2^* \exp\left(-B(1)\left\{v + H_1\Delta S_1^* + H_2\Delta S_2^*\right\}\right)\right].
\end{aligned}$$

- One has to solve a system of nonlinear equations for H_1 and H_2 (numerical methods).

Risk Neutral Computational Approach to the OPP

Risk neutral computational approach

- The previous example shows that the direct approach to solve the **OPP** easily leads to computational difficultites (system of nonlinear equations)
- There is a more efficient approach based on **RNPM**.
- Recall that we want to solve

Optimal Portfolio Problem $(\mathrm{OPP}(v, U))$

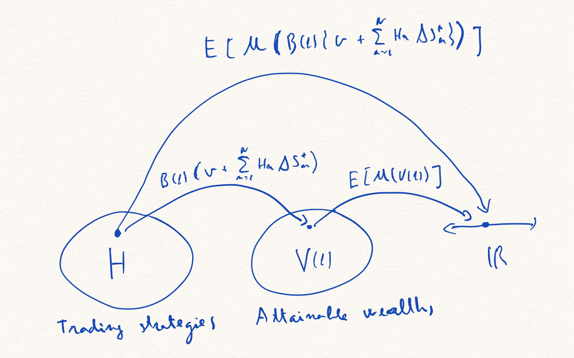
$$\left.\begin{array}{cc} \max & \mathbb{E}\left[U\left(V\left(1\right)\right)\right] \\ \text{subject to} & V\left(0\right) = v \in \mathbb{R}, \\ & H \in \mathbb{R}^{N+1}, \end{array}\right\}$$

Risk neutral computational approach

- The risk neutral computational approach consists in two steps:

Step 1 Maximize $\mathbb{E}\left[U\left(V\left(1\right)\right)\right]$ over the subset of feasible random variables $V\left(1\right)$. That is , determine the optimal terminal wealth $V\left(1\right)$ such that $V\left(0\right)=v$.

Step 2 Given the optimal terminal wealth $V\left(1\right)$, determine a trading strategy H that generates it.



Risk neutral computational approach

Remark 61

- **Step 2** is easy. It boils down to solve a system of linear equations. That is, given $V(1) \in \mathbb{R}^K$, find $H \in \mathbb{R}^{N+1}$ such that

$$\left.\begin{array}{rcl} H_0 B(1) + \sum_{n=1}^{N} H_n S_n(1, \omega_1) & = & V(1, \omega_1) \\ & \vdots & \\ H_0 B(1) + \sum_{n=1}^{N} H_n S_n(1, \omega_1) & = & V(1, \omega_K), \end{array}\right\}$$

- **Step 1** is more challenging and relies on finding a "convenient" feasible region, which we will denote by $\mathbb{W}_v$. Besides this, it is a straightforward optimization problem.

Risk neutral computational approach

- From now on we assume that the model arbitrage free and complete, i.e., $\mathbb{M} = \{Q\}$.
- In this case the set of feasible/attainable wealths is given by

$$\mathbb{W}_v = \left\{ W \in \mathbb{R}^K : \mathbb{E}_Q \left[\frac{W}{B(1)} \right] = v \right\}$$

- Note that, for any trading strategy H with $V(0) = v$ we have, by the risk neutral pricing principle, that

$$\mathbb{E}_Q \left[V(1) / B(1) \right] = V(0) = v.$$

- Conversely, for any $W \in \mathbb{W}_v$ there exists, by the completeness and the risk neutral pricing principle, an H such that $V(0) = v$ and $V(1) = W$.
- The subproblem to solve in **Step 1** is

$$\max_{W \in \mathbb{W}_v} \mathbb{E}\left[U(W) \right].$$

Risk neutral computational approach

- The previous subproblem is a contrained optimization problem, with equality constraints.
- To solve it, we will use the Lagrange multiplier method
- Consider the Lagrangian function

$$\mathcal{L}(W;\lambda) = \mathbb{E}\left[U(W)\right] - \lambda\left(\mathbb{E}_Q\left[\frac{W}{B(1)}\right] - v\right).$$

- Using the state price density $L = Q/P$ we get

$$\begin{aligned}\mathcal{L}(W;\lambda) &= \mathbb{E}\left[U(W)\right] - \lambda\left(\mathbb{E}\left[L\frac{W}{B(1)}\right] - v\right)\\ &= \mathbb{E}\left[U(W) - \lambda\left(L\frac{W}{B(1)} - v\right)\right]\\ &= \sum_{k=1}^{K}\left\{U(W_k,\omega_k) - \lambda L(\omega_k)\frac{W_k}{B(1,\omega_k)} + \lambda v\right\}P(\omega_k),\end{aligned}$$

where $W_k := W(\omega_k)$.

Risk neutral computational approach

- The first order optimality conditions gives

$$0 = \frac{\partial}{\partial \lambda} L(W;\lambda) = -\left(\mathbb{E}_Q\left[\frac{W}{B(1)}\right] - v\right) \Longleftrightarrow \mathbb{E}_Q\left[\frac{W}{B(1)}\right] = v,$$
$$0 = \frac{\partial}{\partial W_k} L(W;\lambda) = \left\{U'(W_k,\omega_k) - \lambda \frac{L(\omega_k)}{B(1,\omega_k)}\right\} P(\omega_k),$$

  where $k = 1, ..., K$.
- Since $U(\cdot,\omega)$ is concave, $U'(\cdot,\omega)$ is decreasing and the inverse of $U'(\cdot,\omega)$ exists, for each $\omega \in \Omega$ fixed.
- Let $I(\cdot,\omega)$ denote the inverse of $U'(\cdot,\omega)$.

Risk neutral computational approach

- A solution $\left(\widehat{W}, \widehat{\lambda}\right)$ of the previous equations is given by $\widehat{W} = I\left(\widehat{\lambda}L/B(1)\right)$, that is,

$$\widehat{W}_k = I\left(\frac{\widehat{\lambda}L(\omega_k)}{B(1,\omega_k)}\right), \qquad k = 1, ..., K,$$

and $\widehat{\lambda}$ is chosen such that

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B(1)}\right] = \mathbb{E}_Q\left[\frac{I\left(\widehat{\lambda}L/B(1)\right)}{B(1)}\right]$$
$$= \sum_{k=1}^{K} \frac{I\left(\widehat{\lambda}L(\omega_k)/B(1,\omega_k)\right)}{B(1,\omega_k)} Q(w_k).$$

- The function I is decreasing and its range will normally include $(0, +\infty)$, so $\widehat{\lambda}$ satisfying the previous equation will exist for $v > 0$.

Risk neutral computational approach

Example 62

- Consider a market with $N=2, K=3, B(0)=1, B(1)=\frac{10}{9}, S_1^*(0)=6, S_2^*(0)=10$, and with payoff matrix

$$S^*(1,\Omega)=\begin{pmatrix} 1 & 6 & 13 \\ 1 & 8 & 9 \\ 1 & 4 & 8 \end{pmatrix}.$$

- We will solve the **OPP** with utility function $U(u)=-e^{-u}$.
- This example is discussed in the smartboard.