9. The Black-Scholes Model

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STK-MAT 3700/4700 An Introduction to Mathematical Finance

November 18, 2021

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Outline

Introduction

Introduction

- The Black-Scholes model is an example of continuous time model for the risky asset prices.

Let us summarize the underlying hypothesis of the Black-Scholes model on the prices of assets.

- The assets are traded continuously and their prices have continuous paths.
- The risk-free interest rate $r \geq 0$ is constant.
- The logreturns of the risky asset S_t are normally distributed:
$$\log\left(\frac{S_t}{S_u}\right) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)(t-u), \sigma^2(t-u)\right).$$
- Moreover, the logreturns are independent from the past and are stationary.
- The model has 3 parameters $\mu \in \mathbb{R}$, $\sigma > 0$ and $S_0 > 0$.

- Let Ω be a set with possibly infinite cardinality.

Definition 1

A σ-algebra $\mathcal{F}$ on Ω is a familly of subsets of Ω satisfying

1. $\Omega \in \mathcal{F}$.
2. If $A \in \mathcal{F}$ then $A^c = \Omega \setminus A \in \mathcal{F}$.
3. If $\{A_n\}_{n\geq 1} \subseteq \mathcal{F}$ then $\bigcup_{n\geq 1} A_n \in \mathcal{F}$.

Definition 2

A pair $(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a σ-algebra on Ω, is called a measurable space.

Definition 3

Given $\mathcal{G}$ a class of subsets of Ω we define $\sigma(\mathcal{G})$ the σ-algebra generated by $\mathcal{G}$ as the smallest σ-algebra containing $\mathcal{G}$, which coincides with the intersection of all σ-algebras containing $\mathcal{G}$.

- In $\mathbb{R}$, we can consider the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, the σ-algebra generated by the open sets.

Definition 4

A probability measure on a measurable space $(\Omega, \mathcal{F})$ is a set function $P : \mathcal{F} \to [0, 1]$ satisfying $P(\Omega) = 1$ and, if $\{A_n\}_{n\geq 1} \subseteq \mathcal{F}$ are pairwise disjoint then

$$P\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} P(A_n).$$

Probability basics

Definition 5

A triple $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is a σ-algebra on Ω and P is a probability measure on $(\Omega, \mathcal{F})$ is called a probability space.

Definition 6

Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ two measurable spaces. A function $X : E_1 \to E_2$ is said to be $(\mathcal{E}_1, \mathcal{E}_2)$-measurable if $X^{-1}(A) \in \mathcal{E}_1$ for all $A \in \mathcal{E}_2$.

Definition 7

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a random variable if it is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable (usually one only write $\mathcal{F}$-measurable).

Definition 8

The σ-algebra generated by a random variable X is the σ-algebra generated by the sets of the form $\{X^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\}$.

Definition 9

The law of a random variable X, denoted by $\mathcal{L}(X)$, is the image measure P_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, that is,

$$P_X(B) = P(X^{-1}B), \quad B \in \mathcal{B}(\mathbb{R}).$$

Definition 10

Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then the expectation of $g(X)$ is defined to be

$$\mathbb{E}\left[g(X)\right] = \int_{\Omega} g \circ X dP = \int_{\mathbb{R}} g dP_X.$$

If $P_X \ll \lambda$, with $\frac{dP_X}{d\lambda} = f_X$ then

$$\mathbb{E}\left[g(X)\right] = \int_{\mathbb{R}} g f_X d\lambda = \int_{\mathbb{R}} g(x) f_X(x) dx.$$

Definition 11

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}[|X|] < \infty$ and $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. The conditional expectation of X given $\mathcal{G}$, denoted by $\mathbb{E}[X|\mathcal{G}]$ is the unique random variable Z satisfying:

1. Z is $\mathcal{G}$-measurable.
2. For all $B \in \mathcal{G}$, we have $\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[Z\mathbf{1}_B]$.

- As Ω does not need to be finite, the structure of the σ-algebras on Ω is not as easy as in the finite case.
- Hence, computing $\mathbb{E}[X|\mathcal{G}]$ is more difficult in general.
- However, $\mathbb{E}[X|\mathcal{G}]$ satisfies the same properties as when Ω was finite: tower law, total expectation, role of the independence,etc...

Definition 12

A (real-valued) stochastic process X indexed by $[0,T]$ is a family of random variables $X = \{X_t\}_{t\in[0,T]}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$.

- We can think of a stochastic process as a function

$$X : \begin{array}{ccc} [0,T]\times\Omega & \longrightarrow & \mathbb{R} \\ (t,\omega) & \mapsto & X_t(\omega) \end{array}.$$

- For every $\omega\in\Omega$ fixed, the process X defines a function

$$X_\cdot(\omega) : \begin{array}{ccc} [0,T] & \longrightarrow & \mathbb{R} \\ t & \mapsto & X_t(\omega) \end{array},$$

  which is called a *trajectory* or a *sample path* of the process.

Stochastic processes

- Hence, we can look at X as a mapping

$$X : \begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R}^{[0,T]} \\ \omega & \mapsto & X_{\cdot}(\omega) \end{array},$$

  where $\mathbb{R}^{[0,T]}$ is the cartesian product of $[0,T]$ copies of $\mathbb{R}$ which is the set of all functions from $[0,T]$ to $\mathbb{R}$. That is, we can see X as a mapping from Ω to a space of functions.
- The canonical construction of a random variable consists on taking $X = Id$ and $(\Omega, \mathcal{F}, P) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$.

Stochastic processes

- For stochastic processes $Y = \{Y_t\}_{t\in[0,T]}$ this procedure is far from trivial. One can consider the measurable space $\left(\mathbb{R}^{[0,T]}, \mathcal{B}(\mathbb{R})^{[0,T]}\right)$ but to find P_Y one needs to do it consistently with the family of finite dimensional laws. (***Kolmogorov Extension Theorem***)
- Moreover, the space $\mathbb{R}^{[0,T]}$ is too big. One often wants to find a realization of the process in a nicer subspace as $C_0([0,T])$. (***Kolmogorov Continuity Theorem***)

Definition 13

A filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is a family of nested σ-algebras, that is, $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s < t$.

Definition 14

A stochastic process $X = \{X_t\}_{t\in[0,T]}$ is $\mathbb{F}$-adapted if X_t is $\mathcal{F}_t$-measurable.

Stochastic processes

Definition 15

A stochastic process $X = \{X_t\}_{t\in[0,T]}$ is a $\mathbb{F}$-martingale if it is $\mathbb{F}$-adapted, $\mathbb{E}\left[|X_t|\right] < \infty$,$t \in [0,T]$ and

$$\mathbb{E}\left[X_t \middle| \mathcal{F}_s\right] = X_s, \quad 0 \le s < t \le T.$$

Definition 16

A stochastic process $X = \{X_t\}_{t\in[0,T]}$ has independent increments if $X_t - X_s$ is independent of $X_r - X_u$, for all $u \le r \le s \le t$.

Definition 17

A stochastic process $X = \{X_t\}_{t\in[0,T]}$ has stationary increments if for all $s \le t \in \mathbb{R}_+$ we have that

$$\mathcal{L}\left(X_t - X_s\right) = \mathcal{L}(X_{t-s}).$$

Brownian motion and related processes

Definition 18

A stochastic process $W = \{W_t\}_{t\in[0,T]}$ is a (standard) Brownian motion if it satisfies

1. W has continuous sample paths P-a.s.,
2. $W_0 = 0$, P-a.s.,
3. W has independent increments,
4. For all $0 \leq s < t \leq T$, the law of $W_t - W_s$ is a $\mathcal{N}(0,(t-s))$.

Definition 19

A stochastic process $W = \{W_t\}_{t \in [0,T]}$ is a $\mathbb{F}$-Brownian motion if it satisfies

1. W has continuous sample paths P-a.s.,
2. $W_0 = 0, P$-a.s.,
3. For all $0 \leq s < t \leq T$, the random variable $W_t - W_s$ is independent of $\mathcal{F}_s$.
4. For all $0 \leq s < t \leq T$, the law of $W_t - W_s$ is a $\mathcal{N}(0, (t - s))$.

Lévy processes

Definition 20

A stochastic process $L = \{L_t\}_{t\in[0,T]}$ is a Lévy process if it satisfies:

1. $L_0 = 0, P$-a.s.,
2. L has independent increments,
3. L has stationary increments, i.e., for all $0 \leq s < t$, the law of $L_t - L_s$ coincides with the law of L_{t-s}.
4. X is stochastically continuous, i.e., $\lim_{s\to t} P(|L_t - L_s| > \varepsilon) = 0, \forall \varepsilon > 0, t \in [0, T]$.

- That L is stochastically continuous does not imply that L has continuous sample paths.
- A Brownian motion is a particular case of Lévy process.
- Useful for modeling stock prices.

Brownian motion with drift and geometric Brownian motion

Definition 21

A stochastic process $Y = \{Y_t\}_{t\in[0,T]}$ is a Brownian motion with drift μ and volatility σ if it can be written as

$$Y_t = \mu t + \sigma W_t, \quad t \in [0, T],$$

where W is a standard Brownian motion.

Definition 22

A stochastic process $S = \{S_t\}_{t\in[0,T]}$ is a geometric Brownian motion (or exponential Brownian motion) with drift μ and volatility σ if it can be written as

$$S_t = \exp\left(\mu t + \sigma W_t\right), \quad t \in [0, T],$$

where W is a standard Brownian motion.

Increments of a geometric Brownian motion

- Note that the paths S are continuous and strictly positive by construction.
- The increments of S are not independent.
- Its relative increments

$$\frac{S_{t_n} - S_{t_{n-1}}}{S_{t_{n-1}}}, \frac{S_{t_{n-1}} - S_{t_{n-2}}}{S_{t_{n-2}}}, \ldots., \frac{S_{t_1} - S_{t_0}}{S_{t_0}},$$

  where $0 \leq t_0 < t_1 < \cdots < t_n \leq T$, are independent and stationary.

- Equivalently,

$$\frac{S_{t_n}}{S_{t_{n-1}}}, \frac{S_{t_{n-1}}}{S_{t_{n-2}}}, ..., \frac{S_{t_1}}{S_{t_0}},$$

  and

$$\log\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right), \log\left(\frac{S_{t_{n-1}}}{S_{t_{n-2}}}\right), ..., \log\left(\frac{S_{t_1}}{S_{t_0}}\right),$$

  where $0 \leq t_0 < t_1 < \cdots < t_n \leq T$, are also independent and stationary.
- Moreover, for $0 \leq s < t \leq T$ the law of S_t / S_s is lognormal with parameters $\mu(t-s)$ and $\sigma^2(t-s)$, that is, the law of

$$\log\left(S_t / S_s\right) \sim \mathcal{N}\left(\mu(t-s), \sigma^2(t-s)\right).$$

The Black-Scholes model

The Black-Scholes model

- The time horizon will be the interval $[0, T]$.
- The price of the riskless asset, denoted by $B = \{B_t\}_{t\in[0,T]}$, is given by $B_t = e^{rt}, 0 \leq t \leq T$.
- The price of the risky asset, denoted by $S = \{S_t\}_{t\in 0,T]}$, is modeled by a continuous time stochastic process satisfying the stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad t \in [0, T],$$
$$S_0 = S_0 > 0.$$

- One can check that the process

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \qquad t \in [0, T],$$

satisfies the previous SDE.
- Therefore, S_t is a geometric Brownian motion with drift $\mu - \frac{\sigma^2}{2}$ and volatility σ.

The Black-Scholes model

- Let $S^* := \left\{ S_t^* = e^{-rt} S_t \right\}_{t \in [0,T]}$.
- Note that $\mathbb{E}\left[e^{\theta Z} \right] = e^{\theta \mu + \frac{\theta^2 \sigma^2}{2}}$ if $Z \sim N\left(\mu, \sigma^2\right)$.
- Then, S^* satisfies

$$\mathbb{E}\left[\left. \frac{S_t^*}{S_s^*} \right| \mathcal{F}_s \right]$$

$$= \mathbb{E}\left[\left. \exp\left(\left(\mu - \frac{\sigma^2}{2} - r \right)(t-s) + \sigma\left(W_t - W_s\right) \right) \right| \mathcal{F}_s \right]$$

$$= \mathbb{E}\left[\exp\left(\left(\mu - \frac{\sigma^2}{2} - r \right)(t-s) + \sigma\left(W_t - W_s\right) \right) \right]$$

$$= \exp\left(\left(\mu - \frac{\sigma^2}{2} - r \right)(t-s) \right) \mathbb{E}\left[\exp\left(\sigma W_{t-s} \right) \right]$$

$$= \exp\left(\left(\mu - \frac{\sigma^2}{2} - r \right)(t-s) + \frac{\sigma^2}{2}(t-s) \right) = e^{(\mu - r)(t-s)}.$$

- Hence, S^* is a martingale under P iff $\mu = r$.
- Does there exist a probability measure Q such that S^* is a martingale under Q?
- The answer is given by Girsanov's theorem. Let Q be given by

$$\frac{dQ}{dP} = \exp\left(-\frac{\mu - r}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right),$$

  then the process

$$\widetilde{W}_t = \frac{\mu - r}{\sigma} t + W_t,$$

  is a Brownian motion under Q.
- Moreover, S^* is a martingale under Q.

Theorem 23 (Risk-neutral pricing principle)

Let X be a contingent claim such that $\mathbb{E}_Q[|X|] < \infty$. Then its arbitrage free price at time t is given by

$$P_X(t) = e^{-r(T-t)}\mathbb{E}_Q[X|\mathcal{F}_t], \qquad 0 \le t \le T.$$

The Black-Scholes pricing formula

Black-Scholes pricing formula

Theorem 24

The prices of European call and a put options are given by

$$C\left(t,S_t\right)=S_t\Phi\left(d_1\left(S_t,T-t\right)\right)-Ke^{-r(T-t)}\Phi\left(d_2\left(S_t,T-t\right)\right),$$
$$P\left(t,S_t\right)=Ke^{-r(T-t)}\Phi\left(-d_2\left(S_t,T-t\right)\right)-S_t\Phi\left(-d_1\left(S_t,T-t\right)\right),$$

respectively, where

$$d_1\left(x,\tau\right)=\frac{\log\left(x/K\right)+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},$$
$$d_2\left(x,\tau\right)=\frac{\log\left(x/K\right)+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},$$

and $\Phi(x)=\int_{-\infty}^{x}\phi(z)dz=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz$. *Note also that* $d_1\left(t,\tau\right)=d_2\left(t,\tau\right)+\sigma\sqrt{\tau}$.

Black-Scholes pricing formula

Proof of Theorem 24.

We will prove the formula for the call option

$$X = (S(T) - K)^{+}.$$

By the risk-neutral valuation principle we know that

$$\begin{aligned} P_X(t) &= e^{-r(T-t)}\mathbb{E}_Q\left[(S(T)-K)^{+}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}_Q\left[\left(\frac{S^*(T)}{S^*(t)}S(t) - e^{-r(T-t)}K\right)^{+}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}_Q\left[\left(\frac{S^*(T)}{S^*(t)}x - e^{-r(T-t)}K\right)^{+}\right]\Bigg|_{x=S(t)} \\ &\triangleq \Gamma(x)|_{x=S(t)}. \end{aligned}$$

Black-Scholes pricing formula

Proof of Theorem 24.

Since

$$\frac{S^*(T)}{S^*(t)} = \exp\left(-\frac{\sigma^2}{2}(T-t) + \sigma\left(\widetilde{W}_T - \widetilde{W}_t\right)\right),$$

and $\widetilde{W}_T - \widetilde{W}_t \sim \mathcal{N}(0, (T-t))$ *under* Q, *we have that*

$$\Gamma(x) = \int_{-\infty}^{+\infty} \phi(z)\left(xe^{-\frac{\sigma^2(T-t)}{2} + \sigma\sqrt{T-t}z} - Ke^{-r(T-t)}\right)^+ dz.$$

Note that

$$xe^{-\frac{\sigma^2(T-t)}{2} + \sigma\sqrt{T-t}z} - Ke^{-r(T-t)} \geq 0 \iff z \geq -d_2(x, T-t).$$

□

Proof of Theorem 24.

Therefore,

$$\begin{aligned}\Gamma(x) &= \int_{-d_2(x,T-t)}^{+\infty} \phi(z)\left(xe^{-\frac{\sigma^2(T-t)}{2}+\sigma\sqrt{T-t}z} - Ke^{-r(T-t)}\right)dz \\ &= x\int_{-d_2(x,T-t)}^{+\infty} \phi(z)\, e^{-\frac{\sigma^2(T-t)}{2}+\sigma\sqrt{T-t}z} dz \\ &\quad - Ke^{-r(T-t)}\int_{-d_2(x,T-t)}^{+\infty} \phi(z)\, dz \\ &= I_1 - I_2.\end{aligned}$$

Using that

$$\phi(z)\, e^{-\frac{\sigma^2(T-t)}{2}+\sigma\sqrt{T-t}z} = \phi\left(z - \sigma\sqrt{T-t}\right),$$

and $d_1(x, T-t) = \sigma\sqrt{T-t} + d_2(x, T-t)$, □

Proof of Theorem 24.

we get

$$\begin{aligned} I_1 &= x \int_{-d_2(x,T-t)}^{+\infty} \phi\left(z - \sigma\sqrt{T-t}\right) dz \\ &= x \int_{-\left(\sigma\sqrt{T-t}+d_2(x,T-t)\right)}^{+\infty} \phi\left(z\right) dz \\ &= x\left(1 - \Phi\left(-d_1\left(x, T-t\right)\right)\right). \end{aligned}$$

On the other hand,

$$I_2 = Ke^{-r(T-t)}\left(1 - \Phi\left(-d_2\left(x, T-t\right)\right)\right).$$

The result follows from the following property of Φ

$$\Phi\left(z\right) = 1 - \Phi\left(-z\right), \qquad z \in \mathbb{R}.$$

The Greeks or sensitivity parameters

- Note that the price of a call option $C(t, S_t)$ actually depends on other variables/parameters

$$C(t, S_t) = C(t, S_t; r, \sigma, K).$$

- The derivatives with respect to these parameters are known as the Greeks and are relevant for risk-management purposes.
- Here, there is a list of the most important:
  - Delta:

$$\Delta = \frac{\partial C}{\partial S}(t, S_t) = \Phi\left(d_1\left(S_t, T-t\right)\right).$$

  - Gamma:

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\Phi'\left(d_1\left(S_t, T-t\right)\right)}{\sigma S_t \sqrt{T-t}} = \frac{\phi\left(d_1\left(S_t, T-t\right)\right)}{\sigma S_t \sqrt{T-t}}.$$

The Greeks or sensitivity parameters

- Theta:

$$\Theta = \frac{\partial C}{\partial t} = -\frac{\sigma S_t \Phi'\left(d_1\left(S_t, T-t\right)\right)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi\left(d_2\left(S_t, T-t\right)\right)$$
$$= -\frac{\sigma S_t \phi\left(d_1\left(S_t, T-t\right)\right)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi\left(d_2\left(S_t, T-t\right)\right).$$

- Rho:

$$\rho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}\Phi\left(d_2\left(S_t, T-t\right)\right).$$

- Vega:

$$\frac{\partial C}{\partial \sigma} = S_t\sqrt{T-t}\Phi'\left(d_1\left(S_t, T-t\right)\right) = S_t\sqrt{T-t}\phi\left(d_1\left(S_t, T-t\right)\right).$$

Convergence of the Cox-Ross-Rubinstein pricing formula to the Black-Scholes pricing formula

Convergence of the CRR formula to the Black-Scholes formula

- We will consider a family of CRR market models indexed by $n \in \mathbb{N}$.
- Partition the interval $[0, T)$ into $[(j-1)\frac{T}{n}, j\frac{T}{n})$, $j = 1...., n$.
- $S_n(j)$ will denote the stock price at time $j\frac{T}{n}$ in the nth binomial model.
- Similarly $B_n(j)$ represents the bank account at time $j\frac{T}{n}$, in the nth binomial model.
- Let $r_n = r\frac{T}{n}$ be the interest rate, where $r > 0$ is the interest rate with continuous compounding, i.e.,

$$\lim_{n\to\infty} (1 + r_n)^n = e^{rT}.$$

- Let $a_n = \sigma\sqrt{\frac{T}{n}}$, where σ is interpreted as the instantaneous volatility.

Convergence of the CRR formula to the Black-Scholes formula

- Set up the *up* and *down* factors by

$$u_n = e^{a_n}(1 + r_n),$$
$$d_n = e^{-a_n}(1 + r_n).$$

- For n sufficiently large $d_n < 1$.
- Moreover, note that $u_n > 1 + r_n$ and that $d_n < 1 + r_n$ for all n.
- Hence, there exists a unique martingale measure Q_n in th nth binomial model for all n.

Convergence of the CRR formula to the Black-Scholes formula

- The parameter q_n in the unique martingale measure in the nth binomial model is

$$q_n = \frac{1 + r_n - d_n}{u_n - d_n} = \frac{1 - e^{-a_n}}{e^{a_n} - e^{-a_n}} = \frac{a_n - \frac{1}{2}a_n^2 + o\left(a_n^2\right)}{2a_n + \frac{1}{3}a_n^3 + o\left(a_n^3\right)}$$
$$= \frac{1}{2} - \frac{1}{4}a_n + o\left(a_n\right),$$

  where $o\left(\delta\right)$ with $\delta > 0$ means $\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$.
- Let $\left\{X_n\left(j\right)\right\}_{j=1,\ldots,n}$ be the Bernoullli r.v. underlying the nth market model. Note that $Q_n\left(X_n\left(j\right) = 1\right) = q_n$ and

$$S_n\left(j\right) = S\left(0\right) u_n^{X_n(1) + \cdots + X_n(j)} d_n^{j - \left(X_n(1) + \cdots + X_n(j)\right)}, \quad j = 1, \ldots, n.$$

Convergence of the CRR formula to the Black-Scholes formula

- The value at time zero of a put option with strike K in the nth binomial market is given by

$$P^n_{\text{Put}}(0) = (1+r_n)^{-n}\,\mathbb{E}_{Q_n}\left[(K - S_n(n))^+\right]$$
$$= \mathbb{E}_{Q_n}\left[\left(\frac{K}{(1+r_n)^n} - S(0)\,e^{Y_n}\right)^+\right],$$

where

$$Y_n = \sum_{j=1}^{n} Y_n(j) = \sum_{j=1}^{n} \log\left(\frac{u_n^{X_n(j)} d_n^{1-X_n(j)}}{(1+r_n)}\right).$$

Convergence of the CRR formula to the Black-Scholes formula

- For n fixed the random variable $Y_n(1), ..., Y_n(n)$ are i.i.d. with

$$
\begin{aligned}
\mathbb{E}_{Q_n}\left[Y_n(j)\right] &= q_n \log\left(\frac{u_n}{1+r_n}\right) + (1-q_n)\log\left(\frac{d_n}{1+r_n}\right) \\
&= \left(\frac{1}{2} - \frac{1}{4}a_n + o(a_n)\right) a_n \\
&\quad + \left(\frac{1}{2} + \frac{1}{4}a_n + o(a_n)\right)(-a_n) \\
&= -\frac{1}{2}a_n^2 + o\left(a_n^2\right), \\
\mathbb{E}_{Q_n}\left[Y_n^2(j)\right] &= a_n^2 + o\left(a_n^2\right), \\
\mathbb{E}_{Q_n}\left[\left|Y_n(j)\right|^m\right] &= o\left(a_n^2\right) \qquad m \geq 3.
\end{aligned}
$$

Definition 25

A sequence $\{X_n\}_{n\geq 1}$ of random variables, possibly defined on different probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$, converges in distribution (or weakly) to X, defined on a probability space $(\Omega, \mathcal{F}, P)$, if

$$\mathbb{E}_{P_n}\left[g\left(X_n\right)\right] \longrightarrow \mathbb{E}_P\left[g\left(X\right)\right], \tag{1}$$

when $n \to +\infty$, for all $g \in C_b(\mathbb{R})$ (space of continuous and bounded functions).

Theorem 26 (Lévy's continuity theorem)

A sequence $\{X_n\}_{n\geq 1}$ of random variables, possibly defined on different probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$, converges in distribution (or weakly) to X, defined on a probability space $(\Omega, \mathcal{F}, P)$, if and only if the sequence of corresponding characteristic functions $\left\{\varphi_{X_n}(\theta) = \mathbb{E}_{P_n}\left[e^{i\theta X_n}\right]\right\}_{n\geq 1}$ converges pointwise to the characteristic function $\varphi_X(\theta) = \mathbb{E}_P\left[e^{i\theta X}\right]$ of X.

Convergence of the CRR formula to the Black-Scholes formula

- Let Y be a random variable defined on some probability space $(\Omega, \mathcal{F}, Q)$ with law $\mathcal{N}\left(-\frac{\sigma^2 T}{2}, \sigma^2 T\right)$.
- Its characteristic function is

$$\varphi_Y(\theta) = \exp\left(-i\theta\frac{\sigma^2 T}{2} - \theta^2\frac{\sigma^2 T}{2}\right).$$

- Since $Y_n(j), \ldots, Y_n(n)$ are i.i.d. we have that

$$\varphi_{Y_n}(\theta) = \mathbb{E}_{Q_n}\left[e^{i\theta Y_n}\right] = \prod_{j=1}^{n}\mathbb{E}_{Q_n}\left[e^{i\theta Y_n(j)}\right] = \mathbb{E}_{Q_n}\left[e^{i\theta Y_n(1)}\right]^n.$$

Convergence of the CRR formula to the Black-Scholes formula

- Expanding the exponential we get

$$\begin{aligned}\varphi_{Y_n}(\theta) &= \left(1 + i\theta\mathbb{E}_{Q_n}\left[Y_n(j)\right] - \frac{\theta^2}{2}\mathbb{E}_{Q_n}\left[Y_n^2(j)\right] + o\left(a_n^2\right)\right)^n \\ &= \left(1 - \left(\frac{i\theta + \theta^2}{2}\right)a_n^2 + o\left(a_n^2\right)\right)^n \\ &= \left(1 - \left(\frac{i\theta + \theta^2}{2}\right)\sigma^2\frac{T}{n} + o\left(1/n\right)\right)^n,\end{aligned}$$

  which converges to $\varphi_Y(\theta)$ as n tends to infinity.
- We can conclude that Y_n converges in distribution to a $\mathcal{N}\left(-\frac{\sigma^2 T}{2}, \sigma^2 T\right)$.

Convergence of the CRR formula to the Black-Scholes formula

- Therefore, since we know that $\{Y_n\}_{n\geq 1}$ converge in law to Y, by applying (1) with $g(x) = \left(Ke^{-rT} - S(0)e^x\right)^+$, we have

$$\lim_{n\to+\infty} \mathbb{E}_{Q_n}\left[\left(Ke^{-rT} - S(0)e^{Y_n}\right)^+\right]$$
$$= \int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \left(Ke^{-rT} - S(0)\exp\left(-\frac{\sigma^2 T}{2} + \sigma\sqrt{T}z\right)\right)^+ dz$$
$$= P_{\text{Put}}(0),$$

where we have used that $Y \sim \mathcal{N}\left(-\frac{\sigma^2 T}{2}, \sigma^2 T\right)$ if and only if $Y = -\frac{\sigma^2 T}{2} + \sigma\sqrt{T}Z$ with $Z \sim \mathcal{N}(0,1)$.

Convergence of the CRR formula to the Black-Scholes formula

- Recall that

$$P_{\mathrm{Put}}^{n}(0) = \mathbb{E}_{Q_n}\left[\left(\frac{K}{(1+r_n)^n} - S(0)\,e^{Y_n}\right)^+\right].$$

- One can check that

$$\left|P_{\mathrm{Put}}^{n}(0) - \mathbb{E}_{Q_n}\left[\left(Ke^{-rT} - S(0)\,e^{Y_n}\right)^+\right]\right| \leq K\left|(1+r_n)^{-n} - e^{-rT}\right|,$$

and, therefore, $P_{\mathrm{Put}}^{n}(0)$ and $\mathbb{E}_{Q_n}\left[\left(Ke^{-rT} - S(0)\,e^{Y_n}\right)^+\right]$ converge to the same limit as n tends to infinity.

Convergence of the CRR formula to the Black-Scholes formula

- Then, we can conclude that

$$\lim_{n\to+\infty} P^n_{\text{Put}}(0) = \lim_{n\to+\infty} \mathbb{E}_{Q_n}\left[\left(Ke^{-rT} - S(0)\,e^{Y_n}\right)^+\right]$$
$$= P_{\text{Put}}(0).$$

- It is easy to check that

$$P_{\text{Put}}(0) = Ke^{-rT}\Phi\left(-d_2\left(S(0), T\right)\right) - S(0)\,\Phi\left(-d_1\left(S(0), T\right)\right),$$

where Φ is the cumulative normal distribution and d_1 and d_2 are the same functions defined in Theorem 24.

Convergence of the CRR formula to the Black-Scholes formula

- By using the put-call parity relationship (on the binomial market and on the Black-Scholes market) one gets that

$$\lim_{n\to+\infty} P^n_{\text{Call}}(0) = \lim_{n\to+\infty}\left(P^n_{\text{Put}}(0) + S(0) - (1+r_n)^{-n}K\right)$$
$$= P_{\text{Put}}(0) + S(0) - e^{-rT}K$$
$$= P_{\text{Call}}(0),$$

where

$$P^n_{\text{Call}}(0) = (1+r_n)^{-n}\,\mathbb{E}_{Q_n}\left[(S(n)-K)^+\right]$$
$$= \mathbb{E}_{Q_n}\left[\left(S(0)\,e^{Y_n} - \frac{K}{(1+r_n)^n}\right)^+\right],$$

and

$$P_{\text{Call}}(0) = S(0)\,\Phi\left(d_1(S(0),T)\right) - Ke^{-rT}\Phi\left(d_2(S(0),T)\right).$$

Convergence of the CRR formula to the Black-Scholes formula

- One can modify the previous arguments to provide the formulas for $P_{\text{Call}}(t)$ and $P_{\text{Put}}(t)$.

Theorem 27

Let $g \in C_b(\mathbb{R})$ and let $X = g(S(T))$ be a contingent claim in the Black-Scholes model. Then the price process of X is given by

$$P_X(t) = \lim_{t \to +\infty} P_X^n(t), \qquad 0 \leq t \leq T,$$

where $P_X^n(t)$,$n \geq 1$ are the price processes of X in the corresponding CRR models.

Convergence of the CRR formula to the Black-Scholes formula

- There exist similar proofs of the previous results using the normal approximation to the binomial law, based on the central limit theorem.
- However, note that here we have a triangular array of random variables $\{Y_n(j)\}_{j=1,\ldots,n}, n \geq 1$. Hence, the result does not follow from the basic version of the central limit theorem.
- Moreover, the asymptotic distribution of Y_n need not be Gaussian if we choose suitably the parameters of the CRR model.
- For instance, if we set $u_n = u$ and $d_n = e^{ct/n}, c < r$ we have that Y_n converges in law to a Poisson random variable.
- This lead to consider the exponential of more general Lévy process as underlying price process for the stock.