8. The Cox-Ross-Rubinstein Model

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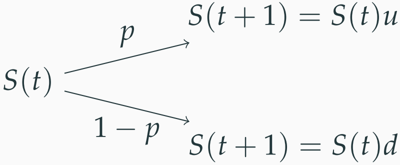
Outline

Introduction

- The Cox-Ross-Rubinstein (**CRR**) market model, also known as the binomial model, is an example of a multi-period market model.
- At each point in time, the stock price is assumed to either go 'up' by a fixed factor u or go 'down' by a fixed factor d .



- Only four parameters are needed to specify the binomial asset pricing model: $u > 1 > d > 0$, $r > -1$ and $S(0) > 0$.
- The real-world probability of an 'up' movement is assumed to be $0 < p < 1$ for each period and is assumed to be independent of all previous stock price movements.

Bernoulli process and related processes

The Bernoulli process

Definition 1

A stochastic process $X = \{X(t)\})_{t\in\{1,...,T\}}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ is said to be a (truncated) **Bernoulli process** with parameter $0 < p < 1$ (and time horizon T) if the random variables $X(1), X(2), ..., X(T)$ are independent and have the following common probability distribution

$$P(X(t) = 1) = 1 - P(X(t) = 0) = p, \qquad t \in \mathbb{N}.$$

- We can think of a Bernoulli process as the random experiment of flipping sequentially T coins.
- The sample space Ω is the set of vectors of zero's and one's of length T. Obviously, $\#\Omega = 2^T$.

The Bernoulli process

- $X(t,\omega)$ takes the value 1 or 0 as ω_t, the t-th component of $\omega \in \Omega$, is 1 or 0, that is, $X(t,\omega) = \omega_t$.
- $\mathcal{F}_t^X$ is the algebra corresponding to the observation of the first t coin flips.
- $\mathcal{F}_t^X = \mathfrak{a}(\pi_t)$ where π_t is a partition with 2^t elements, one for each possible sequence of t coin flips.
- The probability measure P is given by

$$P(\omega) = p^n (1-p)^{T-n},$$

  where ω is any elementary outcome corresponding to n "heads" and $T - n$ "tails".
- Setting this probability measure on Ω is equivalent to say that the random variables $X(1), ..., X(T)$ are independent and identically distributed.

The Bernoulli process

Example

- Consider $T = 3$. Let

$$A_0 = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\},$$
$$A_1 = \{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\},$$
$$A_{0,0} = \{(0,0,0),(0,0,1)\}, \quad A_{0,1} = \{(0,1,0),(0,1,1)\},$$
$$A_{1,0} = \{(1,0,0),(1,0,1)\}, \quad A_{1,1} = \{(1,1,0),(1,1,1)\}.$$

- We have that $\pi_0 = \{\Omega\}$, $\pi_1 = \{A_0, A_1\}$,

$$\pi_2 = \{A_{0,0}, A_{0,1}, A_{1,0}, A_{1,1}\},$$

  $\pi_3 = \{\{\omega\}\}_{\omega\in\Omega}$.
- $\mathcal{F}_t = \mathfrak{a}(\pi_t)$, $t = 0, \ldots, 3$. In particular, $\mathcal{F}_3 = \mathcal{P}(\Omega)$.

The Bernoulli counting process

Definition 2

The **Bernoulli counting process** $N = \{N(t)\}_{t\in\{0,...,T\}}$ is defined in terms of the Bernoulli process X by setting $N(0) = 0$ and

$$N(t,\omega) = X(1,\omega) + \cdots + X(t,\omega), \qquad t \in \{1,...,T\}, \quad \omega \in \Omega.$$

- The Bernoulli counting process is an example of *additive random walk*.
- The random variable $N(t)$ should be thought as the number of heads in the first t coin flips.

The Bernoulli counting process

- Since $\mathbb{E}[X(t)] = p$, Var $[X(t)] = p(1-p)$ and the random variables $X(t)$ are independent, we have

$$\mathbb{E}[N(t)] = tp, \qquad \text{Var}[N(t)] = tp(1-p).$$

- Moreover, for all $t \in \{1, ..., T\}$ one has

$$P(N(t) = n) = \binom{t}{n} p^n (1-p)^{t-n}, \quad n = 0, ..., t,$$

that is, $N(t) \sim Binomial(t, p)$.

The Cox-Ross-Rubinstein model

The CRR market model

- The bank account process is given by $B = \left\{ B(t) = (1+r)^t \right\}_{t=0,\dots,T}$.
- The binomial security price model features 4 parameters: p, d, u and $S(0)$, where $0 < p < 1, 0 < d < 1 < u$ and $S(0) > 0$.
- The time t price of the security is given by

$$S(t) = S(0)\, u^{N(t)} d^{t-N(t)}, \quad t = 1, \dots, T.$$

- The underlying Bernoulli process X governs the *up* and *down* movements of the stock. The stock price moves *up* at time t if $X(t,\omega) = 1$ and moves *down* if $X(t,\omega) = 0$.

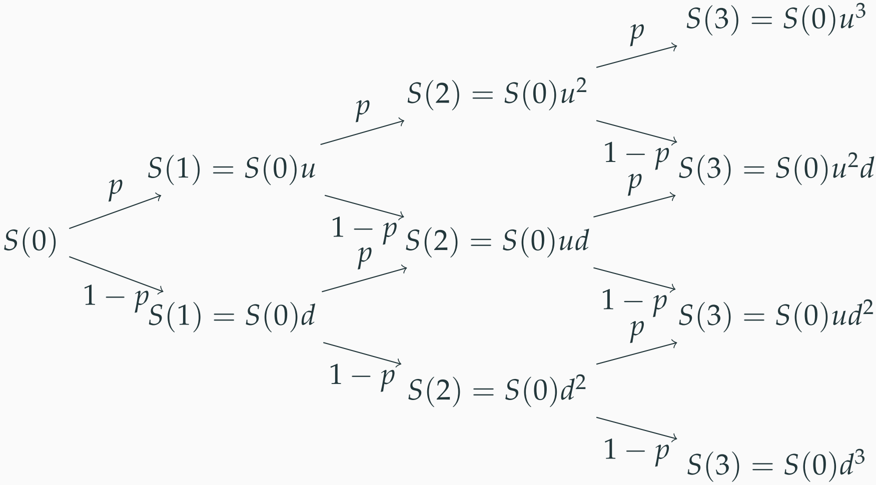
The CRR market model

- The Bernoulli counting process N counts the *up* movements. Before and including time t, the stock price moves up $N(t)$ times and down $t - N(t)$ times.
- The dynamics of the stock price can be seen as an example of a *multiplicative or geometric random walk*.
- The price process has the following probability distribution

$$P\left(S(t) = S(0)\, u^n d^{t-n}\right) = \binom{t}{n} p^n (1-p)^{t-n}, \quad n = 0, ..., t.$$

The CRR market model

- Lattice representation



The CRR market model

- The event $\left\{S\left(t\right) = S\left(0\right) u^n d^{t-n}\right\}$ occurs if and only if exactly n out of the first t moves are *up*. The order of these t moves does not matter.
- At time t, there are 2^t possible sample paths of length t.
- At time t, the price process $S\left(t\right)$ can only take one of $t+1$ possible values.
- This reduction, from exponential to linear in time, in the number of relevant nodes in the lattice is crucial in numerical implementations.

The CRR market model

Example

Consider $T = 2$. Let

$$\Omega = \{(d,d), (d,u), (u,d), (u,u)\}$$
$$A_d = \{(d,d), (d,u)\}, \quad A_u = \{(u,d), (u,u)\}.$$

We have that $\pi_0 = \{\Omega\}, \pi_1 = \{A_d, A_u\}, \pi_2 = \{\{(d,d)\}, \{(d,u)\}, \{(u,d)\}, \{(u,u)\}\}$, and $\mathcal{F}_t = \mathfrak{a}(\pi_t)$, $t = 0, \ldots, 3$. Note that

$$\{S(2) = S(0)ud\} = \{(d,u), (u,d)\} \notin \pi_2.$$

Hence, the lattice representation is NOT the information tree of the model.

Theorem 3

There exists a unique martingale measure in the CRR market model if and only if $d < 1 + r < u$, and is given by

$$Q(\omega) = q^n (1 - q)^{T-n},$$

where ω is any elementary outcome corresponding to n up movements and $T - n$ down movement of the stock and

$$q = \frac{1 + r - d}{u - d}.$$

Corollary 4

If $d < 1 + r < u$, then the CRR model is arbitrage free and complete.

Arbitrage and completeness in the CRR model

Lemma 5

Let Z be a r.v. defined on some prob. space $(\Omega, \mathcal{F}, P)$, with $P(Z = a) + P(Z = b) = 1$ for $a, b \in \mathbb{R}$. Let $\mathcal{G} \subset \mathcal{F}$ be an algebra on Ω. If $\mathbb{E}[Z|\mathcal{G}]$ is constant then Z is independent of $\mathcal{G}$. (Note that the constant must be equal to $\mathbb{E}[Z]$).

Proof of Lemma 5.

Let $A = \{Z = a\}$ and $A^c = \{Z = b\}$. Then for any $B \in \mathcal{G}$

$$\mathbb{E}[Z\mathbf{1}_B] = \mathbb{E}[(a\mathbf{1}_A + b\mathbf{1}_{A^c})\mathbf{1}_B] = aP(A \cap B) + bP(A^c \cap B),$$

and

$$\mathbb{E}[\mathbb{E}[Z]\mathbf{1}_B] = \mathbb{E}[(aP(A) + bP(B))\mathbf{1}_B] = aP(A)P(B) + bP(A^c)P(B).$$

By the definition of cond. expect. we have that $\mathbb{E}[Z\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[Z]\mathbf{1}_B]$. Using that $P(A^c) = 1 - P(A)$ and $P(A^c \cap B) = P(B) - P(A \cap B)$, we get that $P(A \cap B) = P(A)P(B)$ and $P(A^c \cap B) = P(A^c)P(B)$, which yields that $\mathfrak{a}(Z)$ is independent of $\mathcal{G}$. □

Proof of Theorem 3.

Note that $S^*(t) = S(t)(1+r)^{-t}$, $t = 0, \ldots T$. Moreover

$$\begin{aligned}\frac{S(t+1)}{S(t)} &= \frac{S(0)\, u^{N(t+1)} d^{t+1-N(t+1)}}{S(0)\, u^{N(t)} d^{t-N(t)}} \\ &= u^{N(t+1)-N(t)} d^{1-(N(t+1)-N(t))} \\ &= u^{X(t+1)} d^{1-X(t+1)}, \qquad t = 0, \ldots, T-1.\end{aligned}$$

Let Q be another probability measure on Ω.

We impose the martingale condition under Q

$$\mathbb{E}_Q\left[S^*(t+1) \mid \mathcal{F}_t \right] = S^*(t) \Leftrightarrow \mathbb{E}_Q\left[u^{X(t+1)} d^{1-X(t+1)} \,\middle|\, \mathcal{F}_t \right] = 1 + r.$$

Arbitrage free and completeness of the CRR model

Proof of Theorem 3.

This gives

$$(1+r) = \mathbb{E}_Q\left[u^{X(t+1)} d^{1-X(t+1)} \middle| \mathcal{F}_t \right]$$
$$= uQ\left(X(t+1) = 1 \middle| \mathcal{F}_t \right) + dQ\left(X(t+1) = 0 \middle| \mathcal{F}_t \right).$$

In addition,

$$1 = Q\left(X(t+1) = 1 \middle| \mathcal{F}_t \right) + Q\left(X(t+1) = 0 \middle| \mathcal{F}_t \right).$$

Solving the previous equations we get the unique solution

$$Q\left(X(t+1) = 1 \middle| \mathcal{F}_t \right) = \frac{1+r-d}{u-d} = q,$$
$$Q\left(X(t+1) = 0 \middle| \mathcal{F}_t \right) = \frac{u-(1+r)}{u-d} = 1-q.$$

Arbitrage free and completeness of the CRR model

Proof of Theorem 3.

Note that the r.v. $u^{X(t+1)}d^{1-X(t+1)}$ satisfies the hypothesis of Lemma 5 and, therefore, $u^{X(t+1)}d^{1-X(t+1)}$ is independent (under Q) of $\mathcal{F}_t$.

This means that

$$\begin{aligned}(1+r) &= \mathbb{E}_Q\left[u^{X(t+1)}d^{1-X(t+1)}\middle|\,\mathcal{F}_t\right]\\ &= \mathbb{E}_Q\left[u^{X(t+1)}d^{1-X(t+1)}\right]\\ &= uQ\left(X\left(t+1\right)=1\right)+dQ\left(X\left(t+1\right)=0\right),\end{aligned}$$

and we get that

$$\begin{aligned}Q\left(X\left(t+1\right)=1\right) &= Q\left(X\left(t+1\right)=1\middle|\,\mathcal{F}_t\right),\\ Q\left(X\left(t+1\right)=0\right) &= Q\left(X\left(t+1\right)=0\middle|\,\mathcal{F}_t\right).\end{aligned}$$

Proof of Theorem 3.

As the previous unconditional probabilities does not depend on t we obtain that the random variables $X(1), \ldots X(T)$ are identically distributed under Q, i.e. $X(i) = \text{Bernoulli}(q)$. Moreover, for $a \in \{0,1\}^T$ we have that

$$
\begin{aligned}
Q\left(\bigcap_{t=1}^{T}\{X(t)=a_t\}\right) &= \mathbb{E}_Q\left[\prod_{t=1}^{T}\mathbf{1}_{\{X(t)=a_t\}}\right] \\
&= \mathbb{E}_Q\left[\prod_{t=1}^{T-1}\mathbf{1}_{\{X(t)=a_t\}}\mathbb{E}_Q\left[\mathbf{1}_{\{X(T)=a_T\}}\middle|\mathcal{F}_{T-1}\right]\right] \\
&= \mathbb{E}_Q\left[\prod_{t=1}^{T-1}\mathbf{1}_{\{X(t)=a_t\}}Q\left(X(T)=a_T\middle|\mathcal{F}_{T-1}\right)\right] \\
&= \mathbb{E}_Q\left[\prod_{t=1}^{T-1}\mathbf{1}_{\{X(t)=a_t\}}\right]Q\left(X(T)=a_T\right) \\
&= Q\left(\bigcap_{t=1}^{T-1}\{X(t)=a_t\}\right)Q\left(X(T)=a_T\right).
\end{aligned}
$$

Proof of Theorem 3.

Iterating this procedure we get that

$$Q\left(\bigcap_{t=1}^{T}\{X(t)=a_t\}\right)=\prod_{t=1}^{T}Q(X(t)=a_t),$$

and we can conclude that $X(1),...X(T)$ are also independent under Q.

Therefore, under Q, we obtain the same probabilistic model as under P but with $p=q$, that is,

$$Q(\omega)=q^n(1-q)^{T-n}, \qquad n=\sum_{t=1}^{T}\omega_t.$$

The conditions for q are equivalent to $Q(\omega)>0$, which yields that Q is the unique martingale measure. □

Pricing European options in the CRR model

Pricing European options in the CRR model

- By the general theory developed for multiperiod markets we have the following result.

Proposition 6 (Risk Neutral Pricing Principle)

The arbitrage free price process of a European contingent claim X in the CRR model is given by

$$\begin{aligned} P_X(t) &= B(t)\,\mathbb{E}_Q\left[\left.\frac{X}{B(T)}\right|\mathcal{F}_t\right] \\ &= (1+r)^{-(T-t)}\,\mathbb{E}_Q\left[\left.X\right|\mathcal{F}_t\right], \quad t = 0, \ldots, T, \end{aligned}$$

where Q is the unique martingale measure characterized by $q = \frac{1+r-d}{u-d}$.

Pricing European options in the CRR model

- Given g, a non-negative function, define

$$F_{p,g}(t,x) := \sum_{n=0}^{t} \binom{t}{n} p^n (1-p)^{t-n} g\left(xu^n d^{t-n}\right).$$

Proposition 7

Consider a European contingent claim of the form $X = g(S(T))$. Then, the arbitrage free price process $P_X(t)$ is given by

$$P_X(t) = (1+r)^{-(T-t)} F_{q,g}(T-t, S(t)), \qquad t = 0, ..., T,$$

where $q = \frac{1+r-d}{u-d}$.

Pricing European options in the CRR model

Proof of Proposition 7.

Recall that

$$S(t) = S(0)\, u^{N(t)} d^{t-N(t)} = S(0) \prod_{j=1}^{t} u^{X_j} d^{1-X_j}, \quad t = 1, \dots, T.$$

By Proposition 6 we have that

$$\begin{aligned}(1+r)^{(T-t)} P_X(t) &= \mathbb{E}_Q\left[g(S(T)) \middle| \mathcal{F}_t \right] = \mathbb{E}_Q\left[g\left(S(t) \prod_{j=t+1}^{T} u^{X_j} d^{1-X_j} \right) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_Q\left[g\left(s \prod_{j=t+1}^{T} u^{X_j} d^{1-X_j} \right) \right] \Bigg|_{s=S(t)} = F_{q,g}(T-t, S(t)),\end{aligned}$$

where in the last equality we have used that $S(t)$ is $\mathcal{F}_t$-measurable and $X_{t+1}, \dots, X_T$ are independent of $\mathcal{F}_t$.

Note that if X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$ then

$$\mathbb{E}\left[f(X,Y) \middle| \mathcal{G} \right] = \mathbb{E}\left[f(x,Y) \right] \big|_{x=X}.$$

□

Pricing European options in the CRR model

Corollary 8

Consider a European call option with expiry time T and strike price K writen on the stock S. The arbitrage free price $P_C(t)$ of the call option is given by

$$P_C(t) = S(t) \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} \hat{q}^n (1-\hat{q})^{T-t-n} - \frac{K}{(1+r)^{T-t}} \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n},$$

where

$$\hat{n} = \inf\left\{n \in \mathbb{N} : n > \log\left(K/(S(t)\,d^{T-t})\right) / \log(u/d)\right\},$$

and $\hat{q} = \frac{qu}{1+r} \in (0,1)$.

Proof of Corollary 8.

First note that

$$S\left(t\right)u^{n}d^{T-t-n}-K>0\iff n>\log\left(K/\left(S\left(t\right)d^{T-t}\right)\right)/\log\left(u/d\right).$$

Let $g\left(x\right)=\left(x-K\right)^{+}$. If $\hat{n}>T-t$ then $F_{q,g}\left(T-t,S\left(t\right)\right)=0$. If $\hat{n}\leq T-t$, then the formula in Proposition 7 yields

$$\begin{aligned}
&\left(1+r\right)^{T-t}P_{C}\left(t\right)\\
&=F_{q,g}\left(T-t,S\left(t\right)\right)\\
&=\sum_{n=0}^{T-t}\binom{T-t}{n}q^{n}\left(1-q\right)^{T-t-n}\left(S\left(t\right)u^{n}d^{T-t-n}-K\right)^{+}\\
&=\sum_{n=0}^{\hat{n}}\binom{T-t}{n}q^{n}\left(1-q\right)^{T-t-n}0\\
&+\sum_{n=\hat{n}}^{T-t}\binom{T-t}{n}q^{n}\left(1-q\right)^{T-t-n}\left(S\left(t\right)u^{n}d^{T-t-n}-K\right)
\end{aligned}$$

Pricing European options in the CRR model

Proof of Corollary 8.

$$
\begin{aligned}
&= \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n} S(t) u^n d^{T-t-n} \\
&\quad - \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n} K \\
&= S(t) \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} (qu)^n ((1-q)d)^{T-t-n} \\
&\quad - K \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n}.
\end{aligned}
$$

The result follows by defining $\hat{q} = \frac{qu}{1+r}$ and noting that

$$
1 - \hat{q} = \frac{1 + r - qu}{1 + r} = \frac{qu + (1-q)d - qu}{1 + r} = \frac{(1-q)d}{1 + r},
$$

where we have used $qu + (1-q)d = \mathbb{E}_Q\left[u^{X(t+1)} d^{1-X(t+1)}\right] = 1 + r.$ □

Hedging European options in the CRR model

Hedging European options in the CRR model

- Let X be a contingent claim and $P_X = \{P_X(t)\}_{t=0,\dots,T}$ be its price process (assumed to be computed/known).
- As the CRR model is complete we can find a self-financing trading strategy $H = \{H(t)\}_{t=1,\dots,T} = \left\{(H_0(t), H_1(t))^T\right\}_{t=1,\dots,T}$ such that

$$P_X(t) = V(t) = H_0(t)(1+r)^t + H_1(t)S(t), \quad t = 1, \dots, T, \tag{1}$$

$$P_X(0) = V(0) = H_0(1) + H_1(1)S(0).$$

- Given $t = 1, \dots, T$ we can use the information up to (and including) $t-1$ to ensure that H is predictable.
- Hence, at time t, we know $S(t-1)$ but we only know that

$$S(t) = S(t-1)\,u^{X(t)}d^{1-X(t)}.$$

Hedging European options in the CRR model

- Using that $u^{X(t)}d^{1-X(t)} \in \{u, d\}$ we can solve equation (1) uniquely for $H_0(t)$ and $H_1(t)$.
- Making the dependence of P_X explicit on S we have the equations

$$P_X(t, S(t-1)u) = H_0(t)(1+r)^t + H_1(t)S(t-1)u,$$
$$P_X(t, S(t-1)d) = H_0(t)(1+r)^t + H_1(t)S(t-1)d.$$

- The solution for these equations is

$$H_0(t) = \frac{uP_X(t, S(t-1)d) - dP_X(t, S(t-1)u)}{(1+r)^t(u-d)},$$
$$H_1(t) = \frac{P_X(t, S(t-1)u) - P_X(t, S(t-1)d)}{S(t-1)(u-d)}.$$

Hedging European options in the CRR model

- The previous formulas only make use of the lattice representation of the model and not the information tree.

Proposition 9

Consider a European contingent claim $X = g(S(T))$. *Then, the replicating trading strategy* $H = \{H(t)\}_{t=1,\dots,T} = \left\{(H_0(t), H_1(t))^T\right\}_{t=1,\dots,T}$ *is given by*

$$H_0(t) = \frac{uF_{q,g}(T-t, S(t-1)d) - dF_{q,g}(T-t, S(t-1)u)}{(1+r)^T(u-d)},$$

$$H_1(t) = \frac{(1+r)^{T-t}\{F_{q,g}(T-t, S(t-1)u) - F_{q,g}(T-t, S(t-1)d)\}}{S(t-1)(u-d)}.$$

Hedging European options in the CRR model

- Let
$$C\left(\tau,x\right)=\sum_{n=0}^{\tau}\left(\begin{array}{c}\tau\\ n\end{array}\right)q^{n}\left(1-q\right)^{\tau-n}\left(xu^{n}d^{\tau-n}-K\right)^{+}.$$
- Then,
$$P_{C}\left(t\right)=\left(1+r\right)^{-\left(T-t\right)}C\left(T-t,S\left(t\right)\right).$$
- In the following theorem we combine the previous formula and Proposition 9 to find the hedging strategy for a European call option.

Hedging European options in the CRR model

Proposition 10

The replicating trading strategy $H = \{H(t)\}_{t=1,\ldots,T} = \left\{(H_0(t), H_1(t))^T\right\}_{t=1,\ldots,T}$ *for a European call option with strike* K *and expiry time* T *is given by*

$$H_0(t) = \frac{uC(T-t, S(t-1)d) - dC(T-t, S(t-1)u)}{(1+r)^T (u-d)}, .$$

$$H_1(t) = \frac{(1+r)^{T-t}\{C(T-t, S(t-1)u) - C(T-t, S(t-1)d)\}}{S(t-1)(u-d)}.$$

- As $C(\tau, x)$ is increasing in x we have that $H_1(t) \geq 0$, that is, the replicating strategy does not involve short-selling.
- This property extends to any European contingent claim with increasing payoff g.

Hedging European options in the CRR model

- We can also use the value of the contingent claim X and backward induction to find its price process P_X and its replicating strategy H simultaneously.
- We have to choose a replicating strategy $H(T)$ based on the information available at time $T-1$.
- This gives raise to two equations

$$P_X\left(T, S(T-1)u\right) = H_0(T)(1+r)^T + H_1(T)S(T-1)u, \tag{2}$$

$$P_X\left(T, S(T-1)d\right) = H_0(T)(1+r)^T + H_1(T)S(T-1)d. \tag{3}$$

Hedging European options in the CRR model

- The solution is

$$H_0(T) = \frac{uP_X(T, S(T-1)d) - dP_X(T, S(T-1)u)}{(1+r)^T(u-d)},$$

$$H_1(T) = \frac{P_X(T, S(T-1)u) - P_X(T, S(T-1)d)}{S(T-1)(u-d)}.$$

- Next, using that H is self-financing, we can compute

$$P_X(T-1, S(T-1)) = H_0(T)(1+r)^{T-1} + H_1(T)S(T-1),$$

and repeat the procedure (changing T to $T-1$ in equations (2) and (3)) to compute $H(T-1)$.