UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory
Day of examination:	november 2022
Examination hours:	0.00 − 00.00
This problem set consists of 3 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Consider a loan of **1000** to be paid back in 5 equal instalments due at yearly intervals. The instalments include both the interest payable each year calculated at 10% of the current outstanding balance and the repayment of a fraction of the loan. What is the amount of interest included in each instalment? How much of the loan is repaid as part of each instalment?

b (weight 10p)

Suppose that at time $t = 0$, the market price of the underlying asset will be 1000 NOK, the price of a forward contract with a delivery time of one year will be 1080 NOK, under periodic compounding with r = 8%, and short-selling requires a 30% security deposit attracting interest at d = 4%. Is there an arbitrage opportunity? Find the highest rate d for which there is no arbitrage opportunity.

c (weight 10p)

Find the total payout function that depends on the share price of the following securities: 1 call option with a strike price K and 1 put options with a strike price $3K$ are purchased. Construct a graph of the payout function.

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Problem 2

Consider a one-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3\}$, interest rate $r = \frac{1}{10}$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1}$ with prices given by

$$S_1(0) = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} \frac{33}{5} \\ \frac{22}{5} \\ \frac{33}{10} \end{pmatrix}.$$

a (weight 10p)

Find all risk neutral measures in this market. Is this market arbitrage-free? Justify your answer.

b (weight 10p)

Find all contingent claims $X = (X_1, X_2, X_3)^T$ that are attainable in this market. Is this market complete? Justify your answer.

c (weight 10p)

Compute the arbritrage-free price (or prices) for the contingent claims $X = (1, 5, 2)^T$ and $Y = (6, 2, 0)^T$.

Problem 3

Consider a two-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, probability measure $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$, interest rate $r = 0$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1,2}$ with prices given by

$$S_1(0) = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 6 \\ 3 \\ 3 \\ 6 \end{pmatrix}, \quad S_1(2) = \begin{pmatrix} 8 \\ 2 \\ 5 \\ 5 \end{pmatrix}.$$

a (weight 10p)

Find $\{\mathcal{F}_t\}_{t=0,1,2}$, the filtration generated by the price process $S_1 = \{S_1(t)\}_{t=0,1,2}$. Discuss carefully the partitions associated to the price process and how they generate the algebras in the filtration. Calculate $\mathbb{E}\left[S_1(2,\omega) \middle| \mathcal{F}_1\right]$

b (weight 10p)

Find risk neutral probability measure $Q = (Q(\omega_1), Q(\omega_2), Q(\omega_3), Q(\omega_4))$ for given market.

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c (weight 10p)

Consider the following optimal portfolio problem

$$\max_{H\in\mathbb{H}} \mathbb{E}\left[U\left(V\left(2\right)\right)\right]$$
$$\text{subject to } V\left(0\right) = v,$$

where v is a given strictly positive real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U\left(u\right) = 2u^{1/2}$. Compute the optimal attainable wealth, the optimal objective value and the optimal trading strategy.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Suppouse that X, Y, Z are random variables with $X, Y \in \mathcal{F}$. Prove that

$$\mathbb{E}\left[X + YZ|\,\mathcal{F}\right] = X + Y\mathbb{E}\left[Z|\,\mathcal{F}\right].$$

You may use, without having to prove it, that the conditional expectation is a linear operator.

b (weight 10p)

Let $\{X_n,\ n \geq 1\}$ be independent identically distributed random variables such that $\mathbb{E}[X_i] = a, var[X_i] = \sigma^2$, $i \geq 1$. Set $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1$. Find the conditional expectations:

$$\mathbb{E}(X_1 \cdot X_2 \cdot ... \cdot X_n|\mathcal{F}_k).$$

c (weight 10p)

Define what is a martingale with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,....,T}$ under the probability measure P. Let $Z = \{Z(t)\}_{t=0,...,T}$ be a martingale and $H = \{H(t)\}_{t=1,...,T}$ be a predictable process. Prove that the process $G = \{G(t)\}_{t=0,...,T}$ defined by

$$G(0) = 0,$$

$$G(t) = \sum_{u=1}^{t} H(u)(Z(u) - Z(u-1)),\ t = 1, ..., T.$$

is also a martingale.