UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: Tuesday 29. november 2022

Examination hours: 15.00 – 19.00

This problem set consists of 13 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

In general, let assume N years, interest rate r and amount borrowed P. Then

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + ... + \frac{C}{(1+r)^N} = C \cdot \sum_{i=1}^{N} \frac{1}{(1+r)^i}$$

and using the formula $S = \frac{b_1(1-q^n)}{1-q}$ with $b_1 = \frac{1}{1+r}$, q=$\frac{1}{1+r}$ we get

$$P = C \cdot \frac{\frac{1}{1+r}\left(1 - \frac{1}{(1+r)^N}\right)}{1 - \frac{1}{1+r}}$$

Then the formula for each instalment is

$$C = \frac{rP}{1-(1+r)^{-N}}.$$

(Continued on page 2.)

The outstanding balance remaining after $n-1$ instalments is

$$P - \left(\frac{C}{1+r} + \frac{C}{(1+r)^2} + \dots + \frac{C}{(1+r)^{n-1}}\right) =$$

$$= P - \frac{rP}{1-(1+r)^{-N}} \cdot \frac{1-(1+r)^{1-n}}{r} =$$

$$= P \cdot \frac{(1+r)^N - (1+r)^{n-1}}{(1+r)^N - 1}.$$

The interest included in the nth instalment is

$$P\frac{(1+r)^N - (1+r)^{n-1}}{(1+r)^N - 1} r.$$

The capital repaid as part of the nth instalment is

$$C - Pr \cdot \frac{(1+r)^N - (1+r)^{n-1}}{(1+r)^N - 1} =$$

$$= Pr \cdot \frac{(1+r)^N}{(1+r)^N - 1} - Pr \cdot \frac{(1+r)^N - (1+r)^{n-1}}{(1+r)^N - 1} = Pr \cdot \frac{(1+r)^{n-1}}{(1+r)^N - 1}.$$

b (weight 10p)

$S(0) = 1000$ NOK, $F(0,1) = 1070$ NOK.

There will be an arbitrage opportunity.

At time $t = 0$***:***

- Sell short one share for the price $S(0)$, investing 60 % of the proceeds at 8% and the remaining 30% as a security deposit to attract interest at 4%
- Enter a long forward contract with forward price $F(0,1)$

At time $t = 1$***:***

- Collect cash from investments and deposit.
- Buy a share for the price $F(0,1)$ and close the short position in stock.

This leaves an arbitrage profit of 10 NOK:

$$0.6 \cdot S(0) \cdot (1+r) + 0.4 \cdot S(0) \cdot (1+d) > F(0,1);$$

$$0.6 \cdot 1000 \cdot 1.1 + 0.4 \cdot 1000 \cdot 1.05 > 1070.$$

The rate d for the security deposit such that there is no arbitrage opportunity should satisfy

$$0.6 \cdot S(0) \cdot (1+r) + 0.4 \cdot S(0) \cdot (1+d) \leq F(0,1),$$

$$d \leq 0.025.$$

Hence, the highest rate is $d = 2.5\%$.

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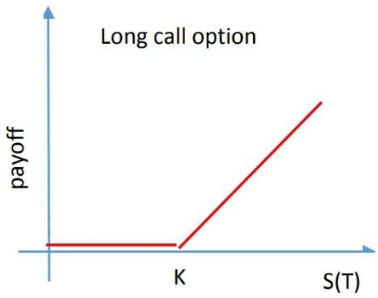
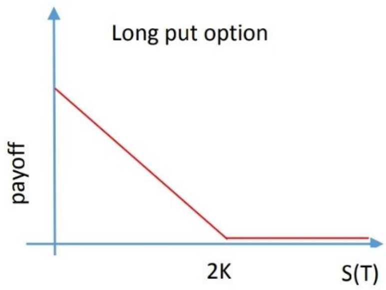
c (weight 10p)

In this strategy you buy a call option with strike k and two put option with the strike $2K$ and the same expiry time T.



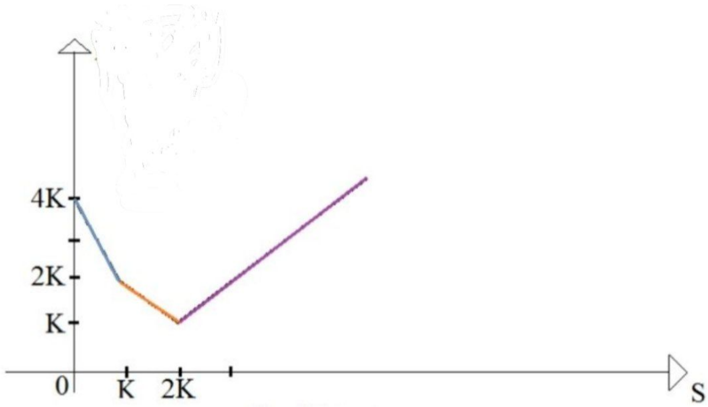


The full payoff function:

$$P(S(T)) = (S(T) - K)^+ + 2(2K - S(T))^+ .$$

In this case, the table of profits is given by

$S(T)$	Profit
$S(T) < K$	$4K - 2S(T)$
$K \leq S(T) \leq 2K$	$3K - S(T)$
$S(T) > 2K$	$S(T) - K$



Problem 2

a (weight 10p)

Let B denote the price process for the bank account. We have that $B(0) = 1$ and $\mathrm{B}(1) = 1 + r = \frac{10}{9}$. The discounted price process for the risky asset is given by

(Continued on page 4.)

$S_1^*(0) = S_1(0)/B(0) = 7$ and $S_1^*(1) = S_1(1)/B(1) = (8,2,5)^T$. A risk neutral probability measure $Q{=}(Q_1, Q_2, Q_3)^T$ must satisfy the following system of equations

$$\begin{aligned} 7 &= S_1^*(0) = \mathbb{E}_Q[S_1^*(1)] = 8Q_1 + 2Q_2 + 5Q_3, \qquad (1)\\ 1 &= Q_1 + Q_2 + Q_3. \end{aligned}$$

We have that $Q_1 = 1 - Q_2 - Q_3$ and substituting this value in equation (1) we obtain

$$1 = 6Q_2 + 3Q_3 \Longleftrightarrow Q_2 = \frac{1-3Q_3}{6}.$$

Moreover,

$$Q_1 = 1 - \frac{1-3Q_3}{6} - Q_3 = \frac{6-1+3Q_3-6Q_3}{6} = \frac{5-3Q_3}{6}.$$

Hence, setting $Q_3 = \lambda$, we get $Q_\lambda = \left(\frac{5-3\lambda}{6}, \frac{1-3\lambda}{6}, \lambda\right)^T$. Finally, as $Q_1 > 0, Q_2 > 0$, and $Q_3 > 0$ we have the following conditions on the parameter λ

$$\begin{aligned} Q_1 &= \frac{5-3\lambda}{6} > 0 \Longleftrightarrow \lambda < \frac{5}{3},\\ Q_2 &= \frac{1-3\lambda}{2} > 0 \Longleftrightarrow \lambda < \frac{1}{3},\\ Q_3 &= \lambda > 0, \end{aligned}$$

which yield that $\lambda \in \left(0, \frac{1}{3}\right)$. Therefore, the set of risk neutral measures $\mathbb{M}$ is given by

$$\mathbb{M} = \left\{ Q_\lambda = \left(\frac{5-3\lambda}{6}, \frac{1-3\lambda}{6}, \lambda\right)^T : 0 < \lambda < \frac{1}{3} \right\}$$

By the first fundamental theorem of asset pricing we know that the market is arbitrage free because the set of risk neutral probability measures is non empty. Alternative parametrizations of $\mathbb{M}$ are

$$\begin{aligned} \mathbb{M} &= \left\{ Q_\lambda = \left(\lambda, \frac{3\lambda-2}{3}, \frac{5-6\lambda}{3}\right)^T, \frac{2}{3} < \lambda < \frac{5}{6} \right\}\\ &= \left\{ Q_\lambda = (2+3\lambda, \lambda, -1-4\lambda)^T, 0 < \lambda < \frac{1}{4} \right\}. \end{aligned}$$

b (weight 10p)

A contingent claim $X = (X_1, X_2, X_3)^T$ is attainable if there exists a portfolio $H = (H_0, H_1)^T$ such that $X = H_0 B(1) + H_1 S_1(1)$. This translates to the following system of equations

$$\begin{aligned} X_1 &= \frac{10}{9}H_0 + \frac{80}{9}H_1,\\ X_2 &= \frac{10}{9}H_0 + \frac{20}{9}H_1,\\ X_3 &= \frac{10}{9}H_0 + \frac{50}{9}H_1. \end{aligned}$$

(Continued on page 5.)

From the first equation we get that $\frac{10}{9}H_0 = X_1 - \frac{80}{9}H_1$. Substituting in the second and third equations we obtain

$$X_3 - X_2 = \frac{30}{9}H_1,$$
$$X_1 - X_3 = \frac{30}{9}H_1.$$

Hence

$$X_1 - X_3 = X_3 - X_2 \Longleftrightarrow X_1 + X_2 - 2X_3 = 0.$$

An alternative way of characterizing the attainable claims, when $\mathbb{M} \neq \emptyset$, is to find $X = (X_1, X_2, X_3)^T$ such that $\mathbb{E}_Q\left[\frac{X}{B(1)}\right]$ does not depend on λ. Hence, since $\lambda \in (0, 1/3)$, we have that

$$\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] = \frac{9}{10}\left\{X_1\lambda + X_2\frac{3\lambda - 2}{3} + X_3\frac{5 - 6\lambda}{3}\right\}$$
$$= \frac{9}{10}\left\{(X_1 + X_2 - 2X_3)\lambda - \frac{2}{3}X_2 + \frac{5}{3}X_3\right\},$$

does not depend on λ (that is, on Q_λ) if and only if $X_1 + X_2 - 2X_3 = 0$.

In addition, by the second fundamental theorem of asset pricing we can conclude that the market is not complete because there are infinitely many risk neutral measures in this market.

c (weight 10p)

The contingent claim $Y = (2, 4, 3)^T$ is attainable because

$$X_1 + X_2 - 2X_3 = 2 + 4 - 6 = 0.$$

Therefore, for this claim the upper and lower hedging price coincide and are equal to

$$\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] = \frac{9}{10}\left\{(X_1 + X_2 - 2X_3)\lambda - \frac{2}{3}X_2 + \frac{5}{3}X_3\right\} =$$
$$= \frac{9}{10}\left\{-\frac{2}{3}X_2 + \frac{5}{3}X_3\right\} = 2.1.$$

Problem 3

a (weight 10p)

We have two-period market with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $P = \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right)^T$, $r = 0$, $B(t) = (1 + r)^t = 1$ and one risky asset $S_1 = \{S_(t)\}_{t=0,1,2}$

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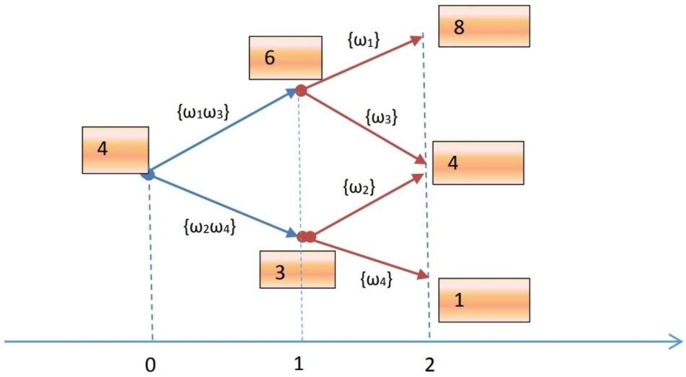

The filtrations are given by

$$\mathcal{F}_0 = \sigma(S_1(0)) = \sigma(\{\Omega\}) = \{\emptyset, \Omega\},$$
$$\mathcal{F}_1 = \sigma(S_1(0), S_1(1)) = \sigma(\{\{\omega_1,\omega_3\},\{\omega_2,\omega_4\}\}) = \{\emptyset,\Omega,\{\omega_1,\omega_3\},\{\omega_2,\omega_4\}\},$$
$$\mathcal{F}_2 = \sigma(S_1(0), S_1(1), S_1(2)) = \sigma(\{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\}) = \mathcal{P}(\Omega),$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω.

$$\mathbb{E}[S_1(2,\omega)|\,\mathcal{F}_1] = \mathbb{E}[S_1(2,\omega)|\,\{\omega_1,\omega_2\}]\,\mathbf{1}_{\{\omega_1,\omega_2\}} + \mathbb{E}[S_1(2,\omega)|\,\{\omega_2,\omega_4\}]\,\mathbf{1}_{\{\omega_2,\omega_4\}} =$$
$$= \left(S_1(2,\omega_1)\frac{p(\omega_1)}{p(\omega_1)+p(\omega_3)} + S_1(2,\omega_3)\frac{p(\omega_3)}{p(\omega_1)+p(\omega_3)}\right)\mathbf{1}_{\{\omega_1,\omega_3\}} +$$
$$+ \left(S_1(2,\omega_2)\frac{p(\omega_2)}{p(\omega_2)+p(\omega_4)} + S_1(2,\omega_4)\frac{p(\omega_4)}{p(\omega_2)+p(\omega_4)}\right)\mathbf{1}_{\{\omega_2,\omega_4\}} =$$

$$= \left(8\cdot\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}} + 4\cdot\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}}\right)\mathbf{1}_{\{\omega_1,\omega_3\}} + + \left(4\cdot\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}} + 1\cdot\frac{\frac{1}{6}}{\frac{1}{3}+\frac{1}{6}}\right)\mathbf{1}_{\{\omega_2,\omega_4\}=}$$
$$= 6\cdot\mathbf{1}_{\{\omega_1,\omega_3\}} + 3\cdot\mathbf{1}_{\{\omega_2,\omega_4\}}$$

b (weight 10p)

By definition a risk neutral probability measure (martingale measure) is a probability measure Q such that

1. $Q(\omega) > 0, \omega \in \Omega$.

(Continued on page 7.)

2. S_n^*, $n = 1, ..., N$ are martingales under Q, that is,

$$\mathbb{E}_Q\left[S_n^*(t+s)|\,\mathcal{F}_t\right] = S_n^*(t), \qquad t, s \geq 0, n = 1, ..., N. \tag{2}$$

It suffices to check (2) for $s = 1$ and $t = 0, ..., T-1$, that is,

$$\mathbb{E}_Q\left[S_n^*(t+1)|\,\mathcal{F}_t\right] = S_n^*(t).$$

If $B(t) = (1+r)^t$, then (2) is equivalent to

$$\mathbb{E}_Q\left[S_n(t+1)|\,\mathcal{F}_t\right] = (1+r)\,S_n(t). \tag{3}$$

We will find $Q = (Q_1, Q_2, Q_3, Q_4)$ satisfying (3) for $t = 0, 1$.
At time $t = 0$:

$$S(0) = \mathbb{E}_Q\left[\frac{S(1)}{1+r}|\mathfrak{F}_0\right] = \mathbb{E}_Q\left[\frac{S(1)}{1+r}\right]$$

For given market we will have

$$6(Q_1 + Q_3) + 3(Q_2 + Q_4) = 4$$

At time $t = 1$:

$$\begin{aligned}(1+r)S(1) = \mathbb{E}_Q[S(2)|\mathfrak{F}_1] = E_Q[S(2)|\omega_1\omega_2]\mathbf{1}_{\{\omega_1,\omega_2\}} + \mathbb{E}_Q[S(2)|\omega_3\omega_4]\mathbf{1}_{\{\omega_3,\omega_4\}} = \\ = \left(S(2,\omega_1)\frac{Q(\omega_1)}{Q(\omega_1,\omega_3)} + S(2,\omega_3)\frac{Q(\omega_3)}{Q(\omega_1,\omega_3)}\right)\mathbf{1}_{\{\omega_1,\omega_3\}} + \\ + \left(S(2,\omega_3)\frac{Q(\omega_3)}{Q(\omega_3,\omega_4)} + S(2,\omega_4)\frac{Q(\omega_4)}{Q(\omega_3,\omega_4)}\right)\mathbf{1}_{\{\omega_3,\omega_4\}} = \\ = \left(8\frac{Q(\omega_1)}{Q(\omega_1,\omega_3)} + 4\frac{Q(\omega_3)}{Q(\omega_1,\omega_3)}\right)\mathbf{1}_{\{\omega_1,\omega_3\}} + \\ + \left(4\frac{Q(\omega_2)}{Q(\omega_2,\omega 4)} + 1\frac{Q(\omega_4)}{Q(\omega_2,\omega_4)}\right)\mathbf{1}_{\{\omega_2,\omega_4\}}.\end{aligned}$$

Note that

$$S(1) = S(1,\omega)\mathbf{1}_{\{\omega_1,\omega_3\}} + S(1,\omega)\mathbf{1}_{\{\omega_2,\omega_4\}},$$
$$S(1) = 6\cdot\mathbf{1}_{\{\omega_1,\omega_3\}} + 3\cdot\mathbf{1}_{\{\omega_2,\omega_4\}}.$$

Hence, we get

$$8Q_1 + 4Q_3 = 6(Q_1 + Q_3),$$
$$4Q_2 + Q_4 = 3(Q_2 + Q_4).$$

By definition RNPM we also have

$$Q_1 + Q_2 + Q_3 + Q_4 = 1.$$

(Continued on page 8.)

As a result, we get the system with four equations

$$Q_1 + Q_2 + Q_3 + Q_4 = 1,$$

$$8Q_1 + 4Q_3 = 6(Q_1 + Q_3),$$

$$4Q_2 + Q_4 = 3(Q_2 + Q_4),$$

$$6(Q_1 + Q_3) + 3(Q_2 + Q_4) = 4$$

We have four variable, four equations and we obtain

$$Q_1 = \frac{1}{6}; \;\; Q_3 = \frac{1}{6};$$

$$Q_2 = \frac{4}{9}; \;\; Q_4 = \frac{2}{9}.$$

An alternative way. Consider three single period market:

- From $t = 0$ to $t = 1$.

  $$6Q(\omega_1\omega_3|\Omega) + 3Q(\omega_2\omega_4|\Omega) = 4;$$

  $$Q(\omega_1\omega_3|\Omega) + Q(\omega_2\omega_4|\Omega) = 1.$$

  Then

  $$Q(\omega_1\omega_3|\Omega) = \frac{1}{3}; \; Q(\omega_2\omega_4|\Omega) = \frac{2}{3}.$$

- From t=1 (S(1)=6) to $t = 2$.

  $$8Q(\omega_1|\omega_1\omega_3) + 4Q(\omega_3|\omega_1\omega_3) = 6;$$

  $$Q(\omega_1|\omega_1\omega_3) + Q(\omega_3|\omega_1\omega_3) = 1.$$

  Then

  $$Q(\omega_1|\omega_1\omega_3) = \frac{1}{2}; \; Q(\omega_3|\omega_1\omega_3) = \frac{1}{2}.$$

- From t=1 (S(1)=3) to $t = 2$

  $$4Q(\omega_2|\omega_2\omega_4) + Q(\omega_4|\omega_2\omega_4) = 3;$$

  $$Q(\omega_2|\omega_2\omega_4) + Q(\omega_4|\omega_2\omega_4) = 1.$$

  Then

  $$Q(\omega_2|\omega_2\omega_4) = \frac{2}{3}; \; Q(\omega_4|\omega_2\omega_4) = \frac{1}{3}.$$

  Hence,

$$Q_1 = Q(\omega_1|\omega_1\omega_3) \cdot Q(\omega_1\omega_3|\Omega) = \frac{1}{6}; \;\; Q_3 = Q(\omega_3|\omega_1\omega_3) \cdot Q(\omega_1\omega_3|\Omega) = \frac{1}{6};$$

$$Q_2 = Q(\omega_2|\omega_2\omega_4) \cdot Q(\omega_2\omega_4|\Omega) = \frac{4}{9}; \;\; Q_4 = Q(\omega_4|\omega_2\omega_4) \cdot Q(\omega_2\omega_4|\Omega) = \frac{2}{9}.$$

(Continued on page 9.)

c (weight 10p)

Since the market is arbitrage free and complete, due to the first and second fundamental theorem of asset pricing. Then, we can use the martingale method to solve the optimal portfolio problem. In this setup, $M = \{Q\}$, the martingale method consists in the following two steps:

1. We first solve the constrained optimization problem
$$\max_{W} \mathbb{E}\left[U(W)\right]$$
$$\text{subject to } \mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v,$$
and obtain the optimal attainable wealth $\widehat{W}$.
2. Given $\widehat{W}$, we find the optimal trading strategy $\widehat{H}$ such that its associated value process $\widehat{V}$ replicates $\widehat{W}$, that is, $\widehat{V}(2) = \widehat{W}$.

The previous constrained problem can be solved using the Lagrange multipliers method. The optimal attainable wealth $\widehat{W}$ is given by

$$\widehat{W} = I\left(\frac{\widehat{\lambda}L}{B(2)}\right),$$

where I is the inverse of $U'(u)$, L is the state-price density vector $L = \frac{Q}{P}$, $B(2)$ is the price of the risk-less asset at time 2 and $\widehat{\lambda}$ is the optimal Lagrange multiplier associated to the constraint $\mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v$. Taking into account that $r = 0$, $U(u) = log(u)$, $P = \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right)^T$ and $Q = \left(\frac{1}{6}, \frac{4}{9}, \frac{1}{6}, \frac{2}{9}\right)^T$,we have that

$$i = U'(u) = \frac{1}{u} \Longleftrightarrow I(i) = \frac{1}{i},$$
$$L = \left(\frac{\frac{1}{6}}{\frac{1}{4}}, \frac{\frac{4}{9}}{\frac{1}{3}}, \frac{\frac{1}{6}}{\frac{1}{4}}, \frac{\frac{2}{9}}{\frac{1}{6}}\right)^T = \left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)^T,$$
$$B(2) = 1,$$

We get that $\widehat{W} = \left(\widehat{\lambda}L\right)^{-1}$. The optimal Lagrange multiplier $\widehat{\lambda}$ satisfies the equation

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B(2)}\right];$$

$$v = \frac{1}{\widehat{\lambda}}\mathbb{E}_Q\left[L^{-1}\right];$$

Therefore, we get

$$\widehat{\lambda} = v^{-1}\mathbb{E}_Q\left[L^{-1}\right].$$

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Note, that

$$\mathbb{E}_Q\left[L^{-1}\right] = \sum_{i=1}^{4} Q_i \cdot \frac{1}{L(\omega_i)} = \sum_{i=1}^{4} P(\omega_i) L(\omega_i) \cdot \frac{1}{L(\omega_i)} = 1.$$

Hence,

$$\widehat{W} = \frac{v}{L} = \left(\frac{3}{2}v; \frac{3}{4}v; \frac{3}{2}v; \frac{3}{4}v\right)^T.$$

and the optimal objective value is given by

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right] = \mathbb{E}\left[log\left(\widehat{W}\right)\right] =$$

$$= \mathbb{E}\left[log\left(\frac{v}{L}\right)\right] = \frac{1}{4}log\left(\frac{3}{2}v\right) + \frac{1}{3}log\left(\frac{3}{4}v\right) + \frac{1}{4}log\left(\frac{3}{2}v\right) + \frac{1}{6}log\left(\frac{3}{4}v\right).$$

Finally, we have to compute the optimal trading strategy $\widehat{H} = \left\{(H_0(t), H_1(t))^T\right\}_{t=1,2}$, that is, a self-financing and predictable process such that its asociated value process V satisfies V(2) $= \widehat{W}$. We first compute the discounted increments of the risky asset

$$\Delta S_1^*(2) = \Delta S_1(2) = (2, 1, -2, -2)^T,$$
$$\Delta S_1^*(1) = \Delta S_1(1) = (2, -1, 2, -1)^T.$$

– For $t = 2$, using that $\widehat{H}$ must be self-financing we have that

$$\frac{\widehat{W}}{B(2)} = \widehat{W} = \widehat{V}^*(1) + \widehat{H}_1(2)\,\Delta S_1^*(2).$$

  * Assuming that $\omega \in \{\omega_1, \omega_3\}$ and the predictability of $\widehat{H}$ we get the equations

$$\frac{3}{2}v = \widehat{W}_1 = \widehat{V}^*(1, \omega_1) + \widehat{H}_1(2, \omega_1) \times 2,$$
$$\frac{3}{2}v = \widehat{W}_3 = \widehat{V}^*(1, \omega_3) + \widehat{H}_1(2, \omega_3) \times (-2),$$
$$\widehat{V}^*(1, \omega_1) = \widehat{V}^*(1, \omega_3),$$
$$\widehat{H}_1(2, \omega_1) = \widehat{H}_1(2, \omega_3),$$

  which, using that $r = 0$, yield

$$\widehat{V}^*(1, \omega_1) = \widehat{V}^*(1, \omega_3) = V(1, \omega_1) = V(1, \omega_3) = \frac{3}{2}v,$$
$$\widehat{H}_1(2, \omega_1) = \widehat{H}_1(2, \omega_3) = 0.$$

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* Assuming that $\omega \in A_{1,2} = \{\omega_1, \omega_3\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned}\frac{3}{4}v = \widehat{W_2} &= \widehat{V}^*(1,\omega_2) + \widehat{H}_1(2,\omega_2) \times 1,\\ \frac{3}{4}v = \widehat{W_4} &= \widehat{V}^*(1,\omega_4) + \widehat{H}_1(2,\omega_4) \times (-2),\\ \widehat{V}^*(1,\omega_1) &= \widehat{V}^*(1,\omega_3),\\ \widehat{H}_1(2,\omega_1) &= \widehat{H}_1(2,\omega_3),\end{aligned}$$

which, using that $r = 0$, yield

$$\begin{aligned}\widehat{V}^*(1,\omega_2) = \widehat{V}^*(1,\omega_4) = V(1,\omega_2) = V(1,\omega_4) &= \frac{3}{4}v,\\ \widehat{H}_1(2,\omega_2) = \widehat{H}_1(2,\omega_4) &= 0.\end{aligned}$$

– For $t = 1$, the predictability assumption yields that $\widehat{H}_1(1)$ is constant. Moreover, using that $\widehat{H}$ must be self-financing we have that $\widehat{V}^*(1) = \widehat{V}^*(0) + \widehat{H}_1(1)\Delta S_1^*(1)$ and we get the following two equations

$$\begin{aligned}\frac{3}{2}v &= \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times 2, \qquad (\text{for } \omega \in \{\omega_1,\omega_3\})\\ \frac{3}{4}v &= \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (-1), \qquad (\text{for } \omega \in \{\omega_2,\omega_4\})\end{aligned}$$

which, using that $r = 0$, yield

$$\widehat{V}^*(0) = V(0) = v, \qquad \widehat{H}_1(1) = \frac{1}{4}v.$$

– Finally we compute $\widehat{H}_0(1)$ and $\widehat{H}_0(2)$ from the definition of value process. We have

$$\widehat{H}_0(1) = \widehat{V}^*(0) - \widehat{H}_1(1)S_1^*(0) = v - \frac{1}{4}v \times 4 = 0,$$

and

$$\begin{aligned}\widehat{H}_0(2,\omega) &= \widehat{V}^*(1,\omega) - \widehat{H}_1(2,\omega)S_1^*(1,\omega)\\ &= \begin{cases} \frac{3}{2}v - 0 \times 6 = \frac{3}{2}v & \text{if } \omega \in \{\omega_1,\omega_3\}\\ \frac{3}{4}v - 0 \times 3 = \frac{3}{4}v & \text{if } \omega \in \{\omega_2,\omega 4\}\end{cases}\end{aligned}$$

Problem 4

a (weight 10p)

If X_1 and X_2 are random variable that are $\mathcal{F}$-measurable, in order to prove that, if Y is $\mathcal{F}$-measurable then

$$\mathbb{E}\left[(X_1Y_1 + X_2Y_2)\middle|\,\mathcal{F}\right] = X_1\mathbb{E}\left[Y_1|\,\mathcal{F}\right] + X_2\mathbb{E}\left[Y_2|\,\mathcal{F}\right].$$

(Continued on page 12.)

we have to prove first that $X_i\mathbb{E}\left[Y_i|\,\mathcal{F}\right], i = 1, 2$ is $\mathcal{F}$-measurable and secondly that

$$\mathbb{E}\left[X_iY_i\mathbf{1}_B\right] = \mathbb{E}\left[X_i\mathbb{E}\left[Y_i|\,\mathcal{F}\right]\mathbf{1}_B\right], \qquad B \in \mathcal{F}. \tag{4}$$

Let $\{A_1, A_2, ..., A_m\}$be the partition that generates $\mathcal{F}$. That $X_i\mathbb{E}\left[Y_i|\,\mathcal{F}\right]$ is a $\mathcal{F}$-measurable random variable follows from the fact that the product of a $\mathcal{F}$-measurable r.v is a $\mathcal{F}$-measurable r.v., because it is constant over the subsets of the partition generating $\mathcal{F}$.

To prove (4), first note that by the linearity of the conditional expectation we can assume that $X_i = \mathbf{1}_{A_i}$ for some $i \in \{1, ..., m\}$ (Recall that an arbitrary $\mathcal{F}$-measurable r.v. is of the form $\sum_{i=1}^m a_i\mathbf{1}_{A_i}$ with $a_i \in \mathbb{R}$). Moreover, for all $B \in \mathcal{F}$ and $i = 1, 2$

$$\begin{aligned}\mathbb{E}\left[X_iY_i\mathbf{1}_B\right] &= \mathbb{E}\left[Y_i\mathbf{1}_{A_i}\mathbf{1}_B\right] = \mathbb{E}\left[Y_i\mathbf{1}_{A_i\cap B}\right] = \mathbb{E}\left[\mathbb{E}\left[Y_i|\,\mathcal{F}\right]\mathbf{1}_{A_i\cap B}\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[Y_i|\,\mathcal{F}\right]\mathbf{1}_{A_i}\mathbf{1}_B\right] = \mathbb{E}\left[\mathbb{E}\left[Y_i|\,\mathcal{F}\right]X_i\mathbf{1}_B\right],\end{aligned}$$

which proves (4). In the third equality we have used the definition of conditional expectation and the fact that $A_i \cap B \in \mathcal{F}$.

b (weight 10p)

We want to find $\mathbb{E}\left[X_1^2 \cdot X_2^2 \cdot ... \cdot X_n^2|\mathcal{F}_k\right]$. As $n > k$, then $X_1, X_2, ..., X_k$ are $\mathcal{F}_k$-measurable, $X_{k+1}, X_{k+2}, ..., X_n$ are independent. So,

$$\begin{aligned}\mathbb{E}\left[X_1^2 \cdot X_2^2 \cdot ... \cdot X_n^2|\mathcal{F}_k\right] &= \mathbb{E}\left[X_1^2 \cdot X_2^2 \cdot ... \cdot X_k^2|\mathcal{F}_k\right] \cdot \mathbb{E}\left[X_{k+1}^2 \cdot X_{k+2}^2 \cdot ... \cdot X_n^2|\mathcal{F}_k\right] =\\ &= X_1^2 \cdot X_2^2 \cdot ... \cdot X_k^2 \cdot (\mathbb{E}[\left[X_{k+1}^2\right] \cdot ... \cdot \mathbb{E}[\left[X_n^2\right]) =\\ &X_1^2 \cdot X_2^2 \cdot ... \cdot X_k^2 \cdot (\sigma^2 + a^2)^{n-k}.\end{aligned}$$

c (weight 10p)

We say that a process $X = \{X(t)\}_{t=0,...,T}$ is a martingale with respect to the filtration $\mathcal{F}$ under the probability measure P if X is $\mathcal{F}$-adapted, that is, $X(t)$ is $\mathcal{F}_t$-measurable for all $t = 0, ..., T$, and

$$\mathbb{E}\left[X(t+s)|\mathcal{F}_t\right] = X(t), \ t, s \geq 0.$$

Or, equivalently,

$$\mathbb{E}[X(t+1)|\mathcal{F}_t] = X(t), t \geq 0.$$

To prove that the process $G = \{G(t)\}_{t=0,...,T}$ defined by

$$G(0) = 0,$$

(Continued on page 13.)

$$G(t) = \sum_{u=1}^{t} H(u)(Z(u) - Z(u-1)),$$

is a martingale first we have to prove that $G(t)$ is $\mathcal{F}$-adapted. First note that if X and Y are $\mathcal{G}$-measurable with respect to an algebra $\mathcal{G}$ on , then XY and $X+Y$ are $\mathcal{G}$-measurable. The process H is predictable and, in particular, adapted to $\mathcal{F}$. The process Z is adapted to $\mathcal{F}$ because it is an $\mathcal{F}$-martingale, moreover $Z(u-1)$ is also $\mathcal{F}_u$-measurable because $\mathcal{F}_{u-1} \subseteq \mathcal{F}_u$. Therefore, $H(u)(Z(u) - Z(u-1))$ is $\mathcal{F}_u$-measurable for $u \leq t$. As $\mathcal{F}$ is a filtration, $\mathcal{F}_u \subseteq \mathcal{F}_t$, and we can conclude that $G(t)$ is $\mathcal{F}_t$-measurable and, hence, G is $\mathcal{F}$-adapted. The result To prove the martingale property, first note that

$$G(t+1) = G(t) + H(t+1)(Z(t+1) - Z(t)).$$

Then,

$$\begin{aligned}
\mathbb{E}[G(t+1)|\mathcal{F}_t] &= \mathbb{E}[G(t) + H(t+1)(Z(t+1) - Z(t))|\mathcal{F}_t] = \\
&= \mathbb{E}[G(t)|\mathcal{F}_t] + \mathbb{E}[H(t+1)(Z(t+1) - Z(t))|\mathcal{F}_t] = \\
&= [G(t) + H(t+1)\mathbb{E}[(Z(t+1) - Z(t))|\mathcal{F}_t] = G(t),
\end{aligned}$$

where in the second equality we have used the linearity of conditional expectation, in the third equality we have used that $H(t+1)$ is $\mathcal{F}_t$-measurable and the property proved in the previous section, and in the fourth equality we have used that Z is a martingale and, therefore,

$$\mathbb{E}[Z(t+1)|\mathcal{F}_t] = Z(t) \Leftrightarrow \mathbb{E}[(Z(t+1) - Z(t))|\mathcal{F}_t] = 0.$$