Chapter 1

The conditional expectation

Definition 1.1

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. A process $M(t), t \geq 0$ is called an $\mathcal{F}_t$-martingale if the following two conditions are satisfied:
1) for any $t \geq 0$, $\mathbb{E}|M(t)| < \infty$,
2) for any $t \geq s \geq 0$:

$$\mathbb{E}(M(t)|\mathcal{F}_s) = M(s) \text{ a.s.}$$

The definition of the martingale in the case when the flow and the family of random variables is indexed by a discrete parameter is given similarly.

Let us recall some properties of the conditional expectation (we assume below that all expectations are finite).

1) If η is $\mathcal{G}$-measurable random variable, then $\mathbb{E}(\eta|\mathcal{G}) = \eta$ a.s.
2) If η is independent of σ-algebra $\mathcal{G}$, then $\mathbb{E}(\eta|\mathcal{G}) = \mathbb{E}\eta$ a.s.
3) If η is measurable with respect to $\mathcal{G}$, then $\mathbb{E}(\eta\tau|\mathcal{G}) = \eta\mathbb{E}(\tau|\mathcal{G})$ a.s.
4) If $\mathcal{G}_1 \subset \mathcal{G}_2$, then $\mathbb{E}(\mathbb{E}(\eta|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(\eta|\mathcal{G}_1)$ a. s.
5) $\mathbb{E}\left(\mathbb{E}(\eta|\mathcal{G})\right) = \mathbb{E}\eta$.
6) Suppose that $\mathbb{E}\xi^2 < \infty$. Then

$$\mathbb{E}\left(\mathbb{E}(\xi|\mathcal{G})\right)^2 \leq \mathbb{E}\xi^2, \quad \text{i.e.,} \quad \|\mathbb{E}(\xi|\mathcal{G})\|_{L_2} \leq \|\xi\|_{L_2},$$

$$\min_{\eta\in L_2(\Omega,\mathcal{G},\mathbb{P})} \mathbb{E}(\xi-\eta)^2 = \mathbb{E}(\xi - \mathbb{E}(\xi|\mathcal{G}))^2.$$

7) Let ξ be a $\mathcal{G}$-measurable random variable, and the random variable η independent of $\mathcal{G}$. Then

$$\mathbb{E}\left(f(\xi,\eta)|\mathcal{G}\right) = m(\xi),$$

where $m(x) = \mathbb{E}\left(f(x,\eta)\right)$. The corresponding expression will be denoted by $\mathbb{E}\left(f(x,\eta)\right)|_{x=\xi}$.

Problem

1. Let $\{\xi_n, \ \ n \geq 1\}$ be independent identically distributed random variables such that $\mathbb{E}[\xi_i] = a, var[\xi_i] = \sigma^2$, $i \geq 1$. Define σ-algebra $\mathcal{F}_n = \sigma(\xi_1, \xi_2, ..., \xi_n)$.

   Calculate

   (a) $\mathbb{E}(\xi_1 + \xi_2 + ... + \xi_n|\mathcal{F}_k)$;

   (b) $\mathbb{E}(\xi_1 \cdot \xi_2 \cdot ... \cdot \xi_n|\mathcal{F}_k)$;

(c) $\mathbb{E}((\xi_1)^2 \cdot (\xi_2)^2 \cdot \ldots \cdot (\xi_n)^2 | \mathcal{F}_k)$;

(d) $\mathbb{E}((\xi_1 + \xi_2 + \ldots + \xi_n)^2 | \mathcal{F}_k)$.

Definition 1.2

Stochastic process $\xi(t), t \geq 0$ is Gaussian if for any $n \geq 1$ and any $0 \leq t_1 \leq \cdots \leq t_n$ the random vector $(\xi(t_1), \ldots, \xi(t_n))$ has the Gaussian distribution.

Definition 1.3

A Gaussian process $W(t), t \geq 0$ is called a standard Wiener process (Brownian motion) if

1. $\mathbb{E}(W(t)) = 0, t \geq 0$;
2. $Cov(W(t), W(s)) = \mathbb{E}(W(t)W(s)) - \mathbb{E}(W(t))\mathbb{E}(W(s)) = min\{s; t\}$, $s, t \geq 0$. (The covariance is the length of the overlapping time period between W(t) and W(s).)

Example 1. Let $\{W(t) : t \geq 0\}$ be a standard Wiener process. Find
a) $\mathbb{E}(2 + 3W(4))^2$;
b) $\mathbb{E}(W(4) - 3W(2))^2$.
Solution. a) Using the properties of expectation we get

$$\mathbb{E}(2 + 3W(4))^2 = \mathbb{E}(4 + 6W(4) + 9W(4)^2) = \mathbb{E}4 + \mathbb{E}(6W(4)) + \mathbb{E}(9W(4)^2)$$

As $\mathbb{E}(W(t)) = 0$ and $\mathbb{E}(W(t))^2 = t$, then

$$\mathbb{E}(2 + 3W(4))^2 = 4 + 6\mathbb{E}(W(4)) + 9\mathbb{E}(W(4)^2) = 4 + 0 + 36 = 40.$$

b) As $\mathbb{E}(W(t)) = 0$, $\mathbb{E}(W(t))^2 = t$, and $\mathbb{E}(W(t)W(s)) = Cov(W(t)W(s)) = min\{s, t\}$, then

$$\mathbb{E}(W(4) - 3W(2))^2 = \mathbb{E}(W^2(4) - 6W(4)W(2) + 9W^2(2)) =$$
$$= \mathbb{E}(W^2(4)) - 6\mathbb{E}(W(4)W(2)) + 9\mathbb{E}(W^2(2)) =$$
$$= 4 - 6 \cdot min\{4, 2\} + 18 = 22 - 12 = 10.$$

Wiener process $W(t), t \geq 0$ is a process with homogeneous increments: random variables $W(v_1) - W(v_2)$ and $W(u_1) - W(u_2)$ have the same distributions for any $v_1, v_2, u_1, u_2 \geq 0$ such that $v_1 - v_2 = u_1 - u_2$.

Example 2. Assume that $W(t)$ is a Wiener process.
Find

$$\mathbb{E}(W(1) \cdot W(2) \cdot W(3) | \mathcal{F}_2).$$

Solution.

$$\mathbb{E}(W(1) \cdot W(2) \cdot W(3) | \mathcal{F}_2) = W(1) \cdot W(2) \cdot \mathbb{E}(W(3) | \mathcal{F}_2) =$$
$$= W(1) \cdot W(2) \cdot \mathbb{E}(W(3) - W(2) + W(2) | \mathcal{F}_2) =$$
$$= W(1) \cdot W(2) \cdot (\mathbb{E}(W(3) - W(2) | \mathcal{F}_2) + \mathbb{E}(W(2) | \mathcal{F}_2)) =$$
$$= W(1)W^2(2).$$

Problem

Let $W(t)$ be an $\mathcal{F}_t$-Winner process.

Find the conditional expectations:

1) $\mathbb{E}(W(1) \cdot W(2) \cdot W(3)|\mathcal{F}_1)$;

2) $\mathbb{E}(W(1) \cdot W(2) \cdot W^2(4)|\mathcal{F}_2)$;

3) $\mathbb{E}(W(1) \cdot W(2) + W(4)|\mathcal{F}_2)$.