Problem 1

Let $\{\xi_n, n \geq 1\}$ be independent identically distributed random variables such that $E[\xi_i] = a$, $var[\xi_i] = \sigma^2$. Define σ-algebra $\mathcal{F}_n = \sigma(\xi_1, \xi_2, \dots, \xi_n)$

Calculate

(a) $E(\xi_1 + \xi_2 + \dots + \xi_n \mid \mathcal{F}_k)$

(b) $E(\xi_1 \cdot \xi_2 \cdot \dots \cdot \xi_n \mid \mathcal{F}_k)$

Solution. (a) $\xi_1, \xi_2, \dots \xi_k$ are $\mathcal{F}_k$-measurable random variable then $E(\xi_1 \mid \mathcal{F}_k) = \xi_1$, $E(\xi_2 \mid \mathcal{F}_k) = \xi_2, \dots, E(\xi_k \mid \mathcal{F}_k) = \xi_k$ a.s., then

$$E(\xi_1 + \xi_2 + \dots + \xi_n \mid \mathcal{F}_k) = E(\xi_1 \mid \mathcal{F}_k) + E(\xi_2 \mid \mathcal{F}_k) + \dots + E(\xi_k \mid \mathcal{F}_k) + E(\xi_{k+1} \mid \mathcal{F}_k) + \dots + E(\xi_n \mid \mathcal{F}_k),$$

where $k < n$.

Note that $\xi_{k+1}, \xi_{k+2}, \dots \xi_n$, $k < n \in N$ are independent of σ-algebra $\mathcal{F}_k \Rightarrow$

$\Rightarrow E(\xi_i \mid \mathcal{F}_k) = E\xi_i$, $i = k+1, \dots n$. a.s.

We have

$$E(\xi_1 + \xi_2 + \dots + \xi_n \mid \mathcal{F}_k) = \underbrace{\xi_1 + \xi_2 + \dots + \xi_k}_{\mathcal{F}_k\text{-measurable}} + E\xi_{k+1} + \dots$$

independent of $\mathcal{F}_k$

$$E(\xi_1+\xi_2+\dots+\xi_n \mid \mathcal{F}_k) = \underbrace{\xi_1+\xi_2+\dots+\xi_k}_{\mathcal{F}_k\text{-measurable}} + E\xi_{k+1} +$$

independent of $\mathcal{F}_k$

$$+ E\xi_{k+2} + \dots + E\xi_{k+1} = \xi_1+\xi_2+\dots+\xi_k +$$

$$+ \underbrace{a+a+\dots+a}_{(n-k)\text{ times}} = \xi_1+\xi_2+\dots+\xi_k+(n-k)a$$

Answer: (a) $= \xi_1+\xi_2+\dots+\xi_k+(n-k)a$, $k<n$

(b) $E(\xi_1\cdot\xi_2\cdots\xi_n \mid \mathcal{F}_k) = E(\xi_1\cdot\xi_2\cdot\xi_3\cdot\ldots\cdot\xi_k\cdot$

$\cdot\,\xi_{k+1}\cdot\ldots\cdot\xi_n \mid \mathcal{F}_k) =$ [Note that ξ_i, $i=1,\dots,n$ is independent, so $E\xi_i\cdot\xi_j = E\xi_i\cdot E\xi_j$] $=$

$$= E(\xi_1\cdot\xi_2\cdot\xi_3\cdot\ldots\cdot\xi_k \mid \mathcal{F}_k)\cdot E(\xi_{k+1}\cdot\xi_{k+2}\cdots\xi_n \mid \mathcal{F}_k)$$

$$= \big(E(\xi_1\mid\mathcal{F}_k)\cdot E(\xi_2\mid\mathcal{F}_k)\cdot\ldots\cdot E(\xi_k\mid\mathcal{F}_k)\big)\cdot$$

$$\cdot\big(E(\xi_{k+1}\mid\mathcal{F}_k)\cdot E(\xi_{k+2}\mid\mathcal{F}_k)\cdot\ldots\cdot E(\xi_n\mid\mathcal{F}_k)\big) =$$

$=$ [Comment: ξ_i, $i=1,\dots,k$ is measurable with respect to $\mathcal{F}_k$ $\Rightarrow$ $E[\xi_i\mid\mathcal{F}_k]=\xi_i$, $i=1,\dots,k$

ξ_i, $i=k+1,\dots,n$ is independent of σ-algebra $\mathcal{F}_k$

$\Rightarrow$ $E[\xi_i\mid\mathcal{F}_k] = E\xi_i$, $i=k+1,\dots,n$.] $=$

$$= \xi_1\cdot\xi_2\cdot\xi_3\cdot\ldots\cdot\xi_k\cdot(E\xi_i)^{n-k} =$$

$$= \xi_1\cdot\xi_2\cdot\xi_3\cdot\ldots\cdot\xi_k\cdot a^{(n-k)}$$

$(n-k)$

Answer: $(b) = \xi_1 \cdot \xi_2 \cdot \xi_3 \cdot \ldots \cdot \xi_k \cdot a^{(n-k)}$

Problem

Let $w(t)$ be an $\mathcal{F}_t$ – Winner process

Find:

$$\mathbb{E}\left(w(1) \cdot w(2) \cdot w(3) \mid \mathcal{F}_2\right)$$

Solution: $w(1), w(2)$ are measurable with respect to σ-algebra $\mathcal{F}_2$. // Recall that if η is G-measurable, then

$$\mathbb{E}(\eta\tau \mid G) = \eta\, \mathbb{E}(\tau \mid G) \quad \text{a.s.} \; //$$

So, $\mathbb{E}\left(w(1) \cdot w(2) \cdot w(3) \mid \mathcal{F}_2\right) =$

$$= w(1) w(2)\, \mathbb{E}\left(w(3) \mid \mathcal{F}_2\right) = w(1) w(2)\, \mathbb{E}\left(w(3) - w(2) + w(2) \mid \mathcal{F}_2\right)$$

$$= w(1) w(2) \Big(\underbrace{\mathbb{E}\left(w(3) - w(2) \mid \mathcal{F}_2\right)}_{= \mathbb{E}(w(3) - w(2)) = \mathbb{E}w(3) - \mathbb{E}w(2) = 0} + \underbrace{\mathbb{E}\left(w(2) \mid \mathcal{F}_2\right)}_{= w(2)} \Big) =$$

$$= w(1) \cdot w^2(2)$$

Answer: $w(1) \cdot w^2(2)$

Note, that we use $\mathbb{E}w(t) = 0$, $t \geq 0$

Find

$$\mathbb{E}\left(w(1) \cdot w(2) w(3) \mid \mathcal{F}_1\right) = w(1)\, \mathbb{E}\left(w(2) w(3) \mid \mathcal{F}_1\right) =$$

$$= w(1)\, \mathbb{E}\left(\left(w(2) - w(1) + w(1)\right) w(3) \mid \mathcal{F}_1\right) =$$

$$E(\eta\tau|G) = \eta E(\tau|G) \text{ a.s. } //$$

So, $E(w(1)\cdot w(2)\cdot w(3)\,|\,\mathcal{F}_2) =$

$$= w(1)w(2)\,E(w(3)\,|\,\mathcal{F}_2) = w(1)w(2)\,E(w(3)-w(2)+w(2)\,|\,\mathcal{F}_2)$$

$$= w(1)w(2)\Big(\underbrace{E(w(3)-w(2)\,|\,\mathcal{F}_2)}_{E(w(3)-w(2)) = Ew(3)-Ew(2)=0} + \underbrace{E(w(2)\,|\,\mathcal{F}_2)}_{w(2)}\Big) =$$

$$= w(1)\cdot w^2(2)$$

Answer: $w(1)\cdot w^2(2)$

Note, that we use $Ew(t)=0,\ t\geq 0$

Find

$$E(w(1)\cdot w(2)w(3)\,|\,\mathcal{F}_1) = w(1)\,E(w(2)w(3)\,|\,\mathcal{F}_1) =$$

$$= w(1)\,E((w(2)-w(1)+w(1))\,w(3)\,|\,\mathcal{F}_1) =$$

$$= w(1)\,E(\underbrace{(w(2)-w(1))\cdot w(3)}_{\text{independent of } \mathcal{F}_1}\,|\,\mathcal{F}_1) + w(1)E(w(1)\cdot w(3)\,|\,\mathcal{F}_1)$$

$$= w(1)\cdot E(\underbrace{(w(2)-w(1))\cdot w(3)}_{\text{indep}}) + w^2(1)\,E(w(3)\,|\,\mathcal{F}_1) =$$

$$= w(1)\cdot \underbrace{E(w(2)-w(1))}_{=0}\cdot \underbrace{Ew(3)}_{=0} + w^2(1)\cdot E(w(3)-w(1)+w(1)\,|\,\mathcal{F}_1)$$

$$= w^2(1)\cdot\big(E(w(3)-w(1))\,|\,\mathcal{F}_1) + E(w(1)\,|\,\mathcal{F}_1)\big) =$$

$$= w^2(1)\cdot \underbrace{E(w(3)-w(1))}_{=0} + w^2(1)\cdot w(1) = w^3(1)$$