STK-MAT3700

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Mandatory assignment

You can typing the solution directly on a computer (for instance with LATEX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number. It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

To pass the assignment you need a score of at least 60p.

Problem 1.

1. (5p.) *How long will it take for a sum of NOK 8000 attracting simple interest to became NOK 8300 if the rate of interest is 9% per year? Also compute the return on this investment.*

   Solutions From formula for simple interest we have

   $$V(t) = V(0)(1 + rt),$$

   or

   $$8300 = 8000(1 + 0.09t) \Leftrightarrow t = \left(\frac{8300}{8000} - 1\right) \cdot (0.09)^{-1},$$

and

$$t = \frac{5}{12}.$$

So, it takes 5 months for the sum of NOK 8000 to become NOK 8300. For daily bases $t \approx 0.4167$ year, which is approximately 153 days.

Further the return on this investment will be

$$K(0,t) := \frac{V(t) - V(0)}{V(0)},$$

$$K(0, 0.4167) = \frac{8300 - 8000}{8000} = 0.0375,$$

i.e. 3.75%.

In the case of simple interest, the return on this investment can be calculated in the next form:

$$K(0,t) = tr = 0.4167 \cdot 0.09 = 0.0375.$$

2. (5p.) *Suppouse that NOK 24500 is deposited at 5.25% compounded continuously. Find the compound amount the interest earned after 6.5 years, effective rate. Find the time required for the original NOK 24500 to grow to NOK 100000.*

Solutions. Continuous compounding is given by $V(t) = V(0)e^{tr}$, and effective rate is given by $r_{ef} = e^r - 1$. So, we get

$$V(6.5) = 24000 \cdot e^{0.00525 \cdot 6.5},$$

$$V(6.5) = 34464.27 \text{ NOK}.$$

After 6.5 years by continuous compounding the amount value will be NOK 34464.27.

Effective rate:

$$r_{ef} = e^{0.0525} - 1,$$

$$r_{ef} = 0.0539,$$

i.e. 5.39%.

The time needed for the original NOK 24500 to grow to NOK 100000 is

$$T = \frac{1}{r} \log\left(\frac{V(T)}{V(0)}\right),$$

$$T \approx 26.79 \; years.$$

3. (5p.) *A loan of NOK 25000 at a rate of 61.5% is paid off in ten years, by paying ten equal installments at the end of every year. How much is each installment?*

   Solutions. We have ordinary annuity certain. Assume that each payment is C. The payments are made at the same time at the end of each interval. Then the present value is

$$PV_{oac} = C \cdot \bar{a_n}|_r,$$

   where $\bar{a_n}|_r = \frac{(1+r)^n - 1}{r(1+r)^n}$. The value of $\bar{a_n}|_r$ will be

$$\bar{a_{10}}|_{0.615} = \frac{(1+0.615)^{10} - 1}{0.615 \cdot (1+0.615)^{10}},$$

$$\bar{a_{10}}|_{0.615} = 1.613.$$

   Then

$$C = PV_{oac} : \bar{a_{10}}|_{0.615} \Leftrightarrow C = 15503.44.$$

   Each installment is NOK 15503.44.

4. (5p.) *Find the present value of an annuity-due of NOK 20000 per quarter for 2 years, if interest is compounded monthly at the nominal rate of 6%.*

   Solutions. This is the situation where the payments are made less frequently than interests is converted. We first calculate the effective rate in the interest per quarter, which is

$$r_{e_q} = \left(1 + \frac{0.06}{12}\right)^3 - 1 = 1.005^3 - 1,$$

   i.e. 1,5%.

   As there are $n = 8$ payments, the required present value is

$$PV_{ad} = C \cdot \ddot{a}_n|_r,$$

   where $\ddot{a}_n|_r = \frac{1-(1+r)^{-n}}{r(1+r)^{-1}}$.

   Now we can calculate $\ddot{a}_n|_r$ with $n = 8$ and $r_{e_q} = 1.005^3 - 1:$

$$\ddot{a}_8|_{r_{e_q}} = \frac{1-(1+r_{e_q})^{-8}}{r_{e_q}(1+r_{e_q})^{-1}} = \frac{1-(1.005)^{-24}}{1-(1.005)^{-3}} \approx 7.5962899977$$

Then

$$PV_{ad} = 20000 \cdot \ddot{a}_0|_{0.015} \approx 151925.8.$$

The Present value of annuity due is NOK 151925.8.

5. (5p.) *The cash flows received for every NOK 10000 of the five-year corporate bond that pays annual coupon payments of 6% represented on the time-line (see. Figure 1.) An investor requires an interest rate of 5 % from this bond. What is the value of this bond?*

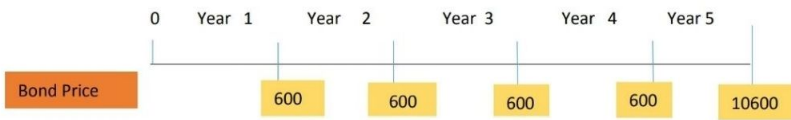

FIGURE 1. The cash flows of bond

Solutions. Consider a bond with face value $F = 10000$ NOK maturing in five years, T = 5, with coupons of $C = 0.06 \cdot 10000 = 600$ NOK paid annually, the last one at maturity. This means a stream of payments of 600, 600, 600, 600, 10600 NOK at the end of each consecutive year.

$$Price = \frac{600}{1+0.05} + \frac{600}{(1+0.05)^2} + \frac{600}{(1+0.05)^3} + \frac{600}{(1+0.05)^4} + \frac{10600}{(1+0.05)^5}.$$

The price of the bond is NOK 10432.94.

6. (5p.) *Find the return on a* 65*-day investment in zero-coupon bonds if* $B(0, 1) = 0.8$.

Solution. Since

$$\frac{1}{1+r} = 0.8$$

the effective rate implied by the bond is 25%. The price of bond after 65 days will be

$$B\left(\frac{65}{365}, 1\right) = B(0,1)(1+r)^{\frac{65}{365}} = 0.8(1+0.25)^{\frac{65}{365}} \approx 0.83243$$

and the return will be

$$K\left(0,\frac{65}{365}\right)=\frac{B\left(\frac{65}{365},1\right)-B(0,1)}{B(0,1)},$$

and

$$K\left(0,\frac{65}{365}\right)\approx 0.04,$$

i.e. 4%.

7. (10p.)Let C^E, P^E, C^A, and P^A denote prices of a European call option, a European put option, an American call option and an American put option, respectively. All of them with expiry time T and the same strike price K. Let $r \geq 0$ be the continuously compounded interest rate. Show that:

   1. If
   $$C^E - P^E - S(0) + Ke^{-rT} < 0,$$
   then you can make a sure risk-less profit.

   2. If
   $$C^A - P^A - S(0) + Ke^{-rT} > 0,$$
   then you can make a sure risk-less profit.

   Solution. To answer this question we will build risk-less strategies with positive profit.

   1) Assume that

   $$C^E - P^E - S(0) + Ke^{-rT} < 0.$$

   At time 0

   - Sell short one share for $S(0)$.
   - Buy one call option for C^E.
   - Write and sell one put option for P^E.
   - Invest $S(0) - C^E + P^E$, (note that by assumption this quantity is positive), risk free at rate r.

   The value of this portfolio is zero.

   At time T:

- Close the money market position, collecting the amount

$$\left(S(0) - C^E + P^E\right) e^{rT}.$$

- Buy one share for K, either by:
  - exercising the call option if $S(T) > K$
  - settling the short position in the put option if $S(T) \leq K$.
  - Close the short selling position by returning the stock to the owner.

This will give a total profit of

$$\left(S(0) - C^E + P^E\right) e^{rT} - K > 0,$$

which is positive by assumption. 2) Assume that

$$C^A - P^A - S(0) + Ke^{-rT} > 0.$$

At time $t = 0$

- Sell a call, buy a put and buy a share, financing the transactions in the money market.

If the American call is exercised at $0 < t \leq T$:

- We get K for the share, closing the short call position.
- We close the money market position.
- We still have the put option which has a non negative value.

The final balance of this strategy is the value of the put at time t and the amount

$$K + \left(C^A - P^A - S(0)\right) e^{rt} =$$
$$= \left(Ke^{-rt} + C^A - P^A - S(0)\right) e^{rt} \geq$$
$$\geq \left(Ke^{-rT} + C^A - P^A - S(0)\right) e^{rt} > 0$$

If the American call is not exercised:

- We sell the share for K, exercising the put option at time T.

- We close the money market position.

The final balance of this strategy gives us

$$K + \left(C^A - P^A - S(0)\right) e^{rT} > 0.$$

8. (10p.) *A call option with strike price of NOK 60 costs NOK 6. A put option with the same strike and expiration date costs NOK 4. Construct a table that shows the profit from a straddle. For what range of stock prices would the straddle lead to a loss? You may assume that the interest rate is zero.*

   Solutions. To make a straddle you buy a call option and a put option with the same strike K. The profit of the straddle as a function of the final price of the stock S_T is given by

$$P(S_T) = (S_T - K)^+ + (K - S_T)^+ - C^E(0) - P^E(0).$$

   In this case, the table of profits is given by

S_T	Profit
$S_T < 60$	$50 - S_T$
$S_T > 60$	$S_T - 70$

   If the final price of the stock lies in the interval $(50, 70)$ the strategy gives a loss.

Problem 2.

1. (5p.) *The price of the stock today is NOK30. Using Table 1 find the price of the shares and draw a graph of changes in the price of the shares.*

Table 1.

	$K(1)$	$K(2)$	$K(3)$
ω_1	5%	5%	-10%
ω_2	-5%	-10%	5%
ω_3	5%	-5%	5%

   Find the coefficients of return $K(0;2)$, $K(0;3)$ and compare the results with the values of $K(1) + K(2)$ and $K(1) + K(2) + K(3)$. Make a conclusion.

Solutions. The return over a single step $[n-1, n]$ is

$$K(n) = K(n-1, n) = \frac{S(n) - S(n-1)}{S(n-1)},$$

which implies that $S(n) = S(n-1)(1 + K(n))$.

So, values prices of shears for ω_1 will be:

$$S(1, \omega_1) = S(0)(1 + K(1, \omega_1)) \Leftrightarrow S(1, \omega_1) = 30(1 + 0.05);$$

$$S(1, \omega_1) = 31.5;$$

$$S(2, \omega_1) = S(1)(1 + K(2, \omega_1)) \Leftrightarrow S(2, \omega_1) = 31.5(1 + 0.05);$$

$$S(2, \omega_1) = 33.075;$$

$$S(3, \omega_1) = S(2)(1 + K(3, \omega_1)) \Leftrightarrow S(3, \omega_1) = 33.075(1 - 0.1);$$

$$S(3, \omega_1) = 29.77.$$

The values prices of shears for ω_2 will be:

$$S(1, \omega_2) = S(0)(1 + K(1, \omega_2)) \Leftrightarrow S(1, \omega_2) = 30(1 - 0.05);$$

$$S(1, \omega_2) = 28.5;$$

$$S(2, \omega_2) = S(1)(1 + K(2, \omega_2)) \Leftrightarrow S(2, \omega_2) = 28.5(1 - 0.1)$$

$$S(2, \omega_2) = 25.65;$$

$$S(3, \omega_2) = S(2)(1 + K(3, \omega_2)) \Leftrightarrow S(3, \omega_2) = 25.65(1 + 0.05)$$

$$S(2, \omega_2) = 26.93;$$

The values prices of shears for ω_3 will be:

$$S(1, \omega_3) = S(0)(1 + K(1, \omega_3)) \Leftrightarrow S(1, \omega_3) = 30(1 + 0.05);$$

$$S(1, \omega_3) = 31.5;$$

$$S(2, \omega_3) = S(1)(1 + K(2, \omega_3)) \Leftrightarrow S(2, \omega_3) = 31.5(1 - 0.05)$$

$$S(2, \omega_3) = 29.93;$$

$$S(3, \omega_3) = S(2)(1 + K(3, \omega_3)) \Leftrightarrow S(3, \omega_3) = 29.93(1 + 0.05)$$

$$S(2, \omega_3) = 31.42.$$

	ω1	ω2	ω3
t=0	30	30	30
t=1	31,5	28,5	31,5
t=2	33,075	25,65	29,933
t=3	31,5	29,925	31,42

We can find the coefficients of return $K(0,2)$ and $K(0,3)$ for each $\omega_i, i = 1,2,3$.

$$K_{\omega_i}(0,2) = \frac{S(2) - S(0)}{S(0)} \text{ and } K_{\omega_i}(0,3) = \frac{S(3) - S(0)}{S(0)}.$$

The values for different type of returns for given data represented in following table:

Scenario	K(1)	K(2)	K(3)	K(1)+K(2)	K(1)+K(2)+K(3)	K(0,2)	K(0,3)
I	5%	5%	-10%	10%	0%	10,25%	-0,80%
II	-5%	-10%	5%	-15%	-10%	-15%	-10,20%
III	5%	-5%	5%	0%	5%	-0,25%	4,70%

It is easy to see that

$$K(0,2) \neq K(1) + K(2) \text{ and } K(0,3) \neq K(1) + K(2) + K(3).$$

The coefficient of return does not have the additivity property.

The precise relationship between consecutive one-step returns and the return over the aggregate period is

$$1 + K(n,m) = (1 + K(n+1))(1 + K(n+2))...(1 + K(m)).$$

2. (5p.) *The price of one bond at time t denoted by $B(t)$. The price of one share at time t denoted by $S(t)$. Let $B(0) = 80$ NOK, $B(1) = 100$ NOK, $S(0) = 30$ NOK and let $S(1) = 40$ NOK with probability 0.6, 25 with probability 0.4. For a portfolio with $x = 10$ shares and $y = 15$ bonds calculate $V(0)$, $V(1)$ and K_V.*

   Solutions. The total wealth of an investor holding x stock shares and y bonds at a time instant $t = 0, 1$ is

$$V(t) = xS(t) + yB(t).$$

   The pair (x, y) is called a portfolio,$V(t)$ being the value of this portfolio. So, for $t = 0$:

$$V(0) = xS(0) + yB(0) \Leftrightarrow V(0) = 10 \cdot 30 + 15 \cdot 80$$

   The value at time 0 of a portfolio with $x = 10$ stock shares and $y = 15$ bonds is $NOK1500$.

   The time 1 value of this portfolio will be

$$V(1) = 10S(1) + 15B(1) = \begin{cases} 10 \cdot 40 + 15 \cdot 100 & \text{with} \quad p = 0.6; \\ 10 \cdot 25 + 15 \cdot 100 & \text{with} \quad p = 0.4. \end{cases}$$

   or

$$V(1) = \begin{cases} 1900 & \text{with} \quad p = 0.6; \\ 1750 & \text{with} \quad p = 0.4. \end{cases}$$

   So the return on the portfolio will be

$$K_V = \frac{V(1) - V(0)}{V(0)} = \begin{cases} 0.267 & \text{with} \quad p = 0.6; \\ 0.167 & \text{with} \quad p = 0.4, \end{cases}$$

   i.e. 26.7 % or 16.7 %.

Problem 3.

Consider a single-period market consisting of a probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, a probability measure $P(\omega) > 0, \omega \in \Omega$, a bank account with $B(0) = 1$, and $r = \frac{1}{9}$, and two risky assets, denoted by $S_1 = \{S_1(t)\}_{t=0,1}$ and $S_2 = \{S_2(t)\}_{t=0,1}$. (See Table 2.)

1. (10p.) Find $S_n^*(0)$, $S_n^*(1, \omega_i)$, for $n = 1, 2$ and $i = 1, 2, 3$. Express magnitudes V^* through the strategy $H = (H_0, H_1, H_2)$ for $\omega_i, i = 1, 2, 3$.

Table 2.

n	$S_n(0)$	$S_n(1)$		
		ω_1	ω_2	ω_3
1	5	20/3	20/3	40/9
2	10	40/3	80/9	80/9

2. (10p.) Define dominant trading strategy and arbitrage opportunity. How are these concepts related?

3. (10p.) Define linear pricing measure and risk neutral pricing measure. How are these concepts related?

Solutions.

1. Given that $B(0) = 1$ and $B(1) = 1 + r = \frac{10}{9}$, where $r = \frac{1}{9}$. Then

$$S_1^*(0) = \frac{S_1(0)}{B(0)} = 5, S_2^*(0) = \frac{S_2(0)}{B(0)} = 10.$$

Using the following equations

$$S_i^*(1) = \frac{S_i(1)}{B(1)}, i = 1, 2$$

we find the shears discounted prices For trading strategy $H = (H_0, H_1, H_2)$

n	$S_n^*(0)$	$S_n^*(1)$		
		ω_1	ω_2	ω_3
1	5	6	6	4
2	10	12	8	8

the discounted value process is given by

$$V^*(t) = H_0 + H_1 S_1^*(t) + H_2 S_2^*(t).$$

Then

$$V^*(1, \omega_1) = H_0 + 6H_1 + 12H_2.$$
$$V^*(1, \omega_2) = H_0 + 6H_1 + 8H_2.$$
$$V^*(1, \omega_3) = H_0 + 4H_1 + 8H_2.$$

2. An **arbitrage oportunity** is a trading strategy $H = (H_0, H_1, H_2)^T$ such that its value process

$$V(t) = H_0 B(t) + H_1 S_1(t) + H_2 S_2(t), \qquad t = 0, 1,$$

satisfies that $V(0) = 0$, $V(1, \omega) \geq 0$ for all $\omega \in \{\omega_1, \omega_2, \omega_3\}$ and $\mathbb{E}[V(1)] > 0$.
A trading strategy $H = (H_0, H_1, H_2)^T$ is **dominant** if there exists another trading strategy $\hat{H} = \left(\hat{H}_0, \hat{H}_1, \hat{H}_2\right)^T$ such that their value processes satisfy that $V(0) = \hat{V}(0)$ and $V(1, \omega) > \hat{V}(1, \omega)$ for all $\omega \in \{\omega_1, \omega_2, \omega_3\}$.
Here

$$V(t) = H_0 B(t) + H_1 S_1(t) + H_2 S_2(t), \qquad t = 0, 1,$$

and

$$\hat{V}(t) = \hat{H}_0 B(t) + \hat{H}_1 S_1(t) + \hat{H}_2 S_2(t), \qquad t = 0, 1.$$

If there exist dominant trading strategies then there exist arbitrage opportunities. However, the existence of arbitrage opportunities do not imply, in general, the existence of dominant trading strategies.

By definition of arbitrage trading strategy we have set of conditions:

$$\left\{\begin{array}{c} V(0) = H_0 + 5H_1 + 10H_2 = 0; \\ V^*(1, \omega_1) = H_0 + 6H_1 + 12H_2 \geq 0; \\ V^*(1, \omega_2) = H_0 + 6H_1 + 8H_2 \geq 0; \\ V^*(1, \omega_3) = H_0 + 4H_1 + 8H_2 \geq 0. \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} H_0 = -5H_1 - 10H_2 = 0; \\ H_1 + 2H_2 \geq 0; \\ H_1 - 2H_2 \geq 0; \\ -H_1 - 2H_2 \geq 0. \end{array}\right.$$

Then

$$H_1 = -2H_2, \;\; H_1 \geq 0, \;\; H_2 \leq 0.$$

Let $H_2 = -1$, then $H_1 = 2$ and $H_0 = 0$. The trading strategy $H = (0, 2, -1)$ is a dominant trading strategy. Really, for this strategy there exists $\omega = \omega_2$, such that $V(1, \omega_2) = \frac{40}{9} > 0$.

3. A **linear pricing measure** is a non-negative vector $\pi = (\pi_1, \pi_2, \pi_3)^T$ such that for every trading strategy H we have that

$$V^*(0) = \sum_{k=1}^{3} \pi_k V^*(1, \omega_k),$$

where $V^* = \left\{ V^*(t) = \frac{V^*(t)}{B(t)} \right\}_{t=0,1}$ is the discounted value process associated to H.
A probability measure Q on Ω is a **risk neutral pricing measure** if $Q(\omega) > 0, \omega \in \{\omega_1, \omega_2, \omega_3\}$ and

$$\mathbb{E}_Q[S_1^*(1)] = S_1^*(0),$$
$$\mathbb{E}_Q[S_2^*(1)] = S_2^*(0),$$

where $S_i^* = \left\{ S_i^* = \frac{S_i(t)}{B(t)} \right\}_{t=0,1}$ for $i = 1, 2$ are the discounted price processes for the two risky assets. A risk neutral pricing measure is always a linear pricing measure. However, a linear pricing measure is a risk neutral measure if only if all its components are strictly positive, i. e., $\pi > 0$.

Let us find linear pricing measure $\pi = (\pi_1, \pi_2, \pi_3)$. Set of conditions for a linear pricing measure have the next form:

$$\begin{cases} \pi_1 + \pi_2 + \pi_3 = 1; \\ 6\pi_1 + 6\pi_2 + 4\pi_3 = 5; \\ 12\pi_1 + 8\pi_2 + 8\pi_3 = 10. \end{cases}$$

The solution of this system is $\pi = (\frac{1}{2}, 0, \frac{1}{2})$. If there exist a linear pricing measure, then doesn't exist dominant trading strategy. Conclusion. The given financial market has arbitrage opportunities. For this market exist a linear pricing measure, do not exist dominant trading strategies.

Problem 4.

Consider a single-period market consisting of a probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, a probability measure $P(\omega) > 0, \omega \in \Omega$, a bank account with $B(0) = 1$, and $B(1) = 1 + r$, where $r \geq 0$ is a given interest rate, and one risky asset, denoted by $S_1 = \{S_1(t)\}_{t=0,1}$,

$$S_1(0) = 3, \qquad S_1(1, \omega) = \begin{cases} 4 & \text{if} \quad \omega = \omega_1 \\ 3 & \text{if} \quad \omega = \omega_2 \\ 2 & \text{if} \quad \omega = \omega_3 \end{cases}.$$

(a) (10p.) Determine the risk-neutral probability measures. Is the market free of arbitrage? Discuss the result in terms of the possible values of r.

(b) (10p.) What is the definition of a complete market? Is the market complete? Determine (characterize) the attainable claims. Discuss the result in terms of the possible values of r.

(c) (10p.) Set $r = 1/6$. Consider the contingent claim $X = (4, 7/2, 4)^T$. Determine the arbitrage-free prices of X.

(d) (10p.) Set $r = 1/6$. Assume that in the market is introduced a new risky asset S_2 with $S_2(0) = \frac{6}{7}$. Give conditions on $S_2(1) = (S_2(1,\omega_1), S_2(1,\omega_2), S_2(1,\omega_3))^T$ such that the extended market is complete. Check if $S_2(1) = (3/2, 1/3, 1/4)^T$ completes the market and, if this is the case, give the unique risk neutral measure.

Solution. We have that $S_1^*(0) = S_1(0) = 3$ and

$$S_1^*(1,\omega) = \begin{cases} 4(1+r)^{-1} & \text{if} \quad \omega = \omega_1 \\ 3(1+r)^{-1} & \text{if} \quad \omega = \omega_2 \\ 2(1+r)^{-1} & \text{if} \quad \omega = \omega_3 \end{cases},$$

$$\Delta S_1^*(\omega) = \begin{cases} (1-3r)(1+r)^{-1} & \text{if} \quad \omega = \omega_1 \\ -3r(1+r)^{-1} & \text{if} \quad \omega = \omega_2 \\ -(1+3r)(1+r)^{-1} & \text{if} \quad \omega = \omega_3 \end{cases}.$$

$Q = (Q_1, Q_2, Q_3)^T$ is a risk neutral pricing measure if and only if Q is a probability measure and $\mathbb{E}_Q[\Delta S_1^*] = 0$, that is, the following equations are satisfied

$$\begin{aligned} (1-3r)Q_1 - 3rQ_2 - (1+3r)Q_3 &= 0, \\ Q_1 + Q_2 + Q_3 &= 1, \\ Q_1 > 0, Q_2 > 0, Q_3 &> 0. \end{aligned}$$

These equations are equivalent to

$$\begin{aligned} Q_3 &= 1 - Q_1 - Q_2, \\ 1 + 3r &= 2Q_1 + Q_2, \\ Q_1 > 0, Q_2 &> 0, Q_3 > 0. \end{aligned}$$

Solving the second equation for Q_2 we get that

$$\begin{aligned} Q_2 &= 1 + 3r - 2Q_1, \\ Q_3 &= 1 - Q_1 - 1 - 3r + 2Q_1 = -3r + Q_1, \end{aligned}$$

Let $Q_1 = \lambda$, then $Q_2 = 1 + 3r - 2\lambda$, $Q_3 = -3r + \lambda$. By definition $0 < Q_i < 1$. It mean that $0 < \lambda < 1$, and

$$0 < 1 + 3r - 2\lambda < 1,$$

$$0 < -3r + \lambda < 1.$$

And combining the previous expressions we get that, for $r \in (0, 1/3)$ the set of risk neutral probability measures is given by

$$Q_\lambda = (\lambda, 1 + 3r - 2\lambda, -3r + \lambda)^T, \lambda \in \left(3r, \frac{1+3r}{2}\right).$$

By the first fundamental theorem of asset pricing the market is arbitrage free if the set of risk neutral pricing measures is non-empty. Hence, if $r \geq 1/3$ there are arbitrage opportunities because risk neutral probability measures do not exist. And if $0 < r < 1/3$ then the market is arbitrage free because exsist RNPM: $Q_\lambda = (\lambda, 1 + 3r - 2\lambda, -3r + \lambda)^T$.

4. A single period market model is complete if any contingent claim X is attainable, that is, if for any contingent claim X there is a trading strategy H such that its value process at time 1 coincides with X, that is,

$$X = V(1) = H_0 B(1) + \sum_{n=1}^{N} H_n S_n(1).$$

The set of attainable claims $X = (X_1, X_2, X_3)^T$ is given by the equations

$$X_1 = (1 + r) H_0 + 4H_1,$$
$$X_2 = (1 + r) H_0 + 3H_1,$$
$$X_3 = (1 + r) H_0 + 2H_1.$$

From the first equation we get that $(1 + r) H_0 = X_1 - 4H_1$ and substituting in the second and third equation this expression for $(1 + r) H_0$ we obtain

$$X_2 - X_1 = -H_1,$$
$$X_3 - X_1 = -2H_1,$$

which is equivalent to

$$X_3 - 2X_2 + X_1 = 0. \tag{1}$$

Hence, the attainable claims are characterized by equation (1). This implies that the market is not complete, regardless of the value of r.

5. If we set $r = 1/6$ we have that $B(1) = 7/6$ and

$$Q_\lambda = \left(\lambda, \frac{3}{2} - 2\lambda, \lambda - \frac{1}{2}\right)^T, \lambda \in (1/2, 3/4).$$

The contingent claim $X = (4, 7/2, 4)^T$ does not satisfy the equation (1). Therefore, the claim X is not attainable and there is an interval of arbitrage free prices $[V_-(X), V_+(X)]$, where $V_-(X)$ is the lower hedging price of X and $V_+(X)$ is the upper hedging price of X. Moreover, we know that

$$\begin{aligned} V_-(X) &= \frac{6}{7} \inf_{\lambda \in (1/2, 3/4)} \{\mathbb{E}_{Q_\lambda}[X]\} \\ &= \frac{6}{7} \inf_{\lambda \in (1/2, 3/4)} \left\{4\lambda + \frac{7}{2}\left(\frac{3}{2} - 2\lambda\right) + 4\left(\lambda - \frac{1}{2}\right)\right\} \\ &= \frac{6}{7} \inf_{\lambda \in (1/2, 3/4)} \left\{\lambda + \frac{13}{4}\right\} = \frac{6}{7}\left\{\frac{1}{2} + \frac{13}{4}\right\} \\ &= \frac{45}{14} \simeq 3.2143 \end{aligned}$$

and

$$\begin{aligned} V_+(X) &= \frac{6}{7} \sup_{\lambda \in (1/2, 3/4)} \{\mathbb{E}_{Q_\lambda}[X]\} \\ &= \frac{6}{7} \sup_{\lambda \in (1/2, 3/4)} \left\{\lambda + \frac{13}{4}\right\} = \frac{6}{7}\left\{\frac{3}{4} + \frac{13}{4}\right\} \\ &= \frac{24}{7} \simeq 3.4286. \end{aligned}$$

6. Let $X = (X_1, X_2, X_3)^T$ be an arbitrary contingent claim. The enlarged payoff matrix is given by

$$S(1, \Omega) = \begin{pmatrix} \frac{7}{6} & 4 & S_2(1, \omega_1) \\ \frac{7}{6} & 3 & S_2(1, \omega_2) \\ \frac{7}{6} & 2 & S_2(1, \omega_3) \end{pmatrix}.$$

The enlarged market is complete iff $S(1,\Omega)H = X$ always have a solution. As this is a system with 3 equations and 3 unknowns, it has a solution iff

$$\det(S(1,\Omega)) \neq 0.$$

We have that

$$\det(S(1,\Omega)) = \begin{vmatrix} \frac{7}{6} & 4 & S_2(1,\omega_1) \\ \frac{7}{6} & 3 & S_2(1,\omega_2) \\ \frac{7}{6} & 2 & S_2(1,\omega_3) \end{vmatrix} =$$

$$= \frac{7}{6}\left\{-S_2(1,\omega_1) + 2S_2(1,\omega_2) - S_2(1,\omega_3)\right\}.$$

Therefore, the market is complete iff

$$-S_2(1,\omega_1) + 2S_2(1,\omega_2) - S_2(1,\omega_3) \neq 0. \tag{2}$$

However, this condition may be fulfilled and still have a market with arbitrages. That is, a market with no risk neutral probability measures (actually, if there are risk neutral probability measures there can only be one, by the SFTAP). In order to ensure the existence of risk neutral probability measure for BOTH assets, we will find $\lambda \in (1/2, 3/4)$ for which such that Q_λ with $r = \frac{1}{6}$ are still risk neutral probability measures in the extended market. That is, the set of Q_λ, with $r = \frac{1}{6}$ and $\lambda \in (1/2, 3/4)$ such that

$$\mathbb{E}_Q\left[S_2^*(1)\right] = S_2^*(0) = S_2(0) = 6/7,$$

This translates to the following equation

$$S_2(1,\omega_1)\lambda + S_2(1,\omega_2)\left(\frac{3}{2} - 2\lambda\right) + S_2(1,\omega_3)\left(\lambda - \frac{1}{2}\right) = 1,$$

with $\lambda \in (1/2, 3/4)$. From this equation we get that

$$\lambda = \frac{1 - \frac{3}{2}S_2(1,\omega_2) + \frac{1}{2}S_2(1,\omega_3)}{S_2(1,\omega_1) - 2S_2(1,\omega_2) + S_2(1,\omega_3)}, \tag{3}$$

which combined with $\lambda \in (1/2, 3/4)$ yields the following inequalities that ensure that the set of risk neutral measures in the enlarged market is non-empty

$$2 - S_2(1,\omega_2) - S_2(1,\omega_1) > 0, \tag{4}$$

$$4 - 3S_2(1,\omega_1) - S_2(1,\omega_3) < 0. \tag{5}$$

The conditions given by equations (2), (4), (5) ensure that the enlarged market is arbitrage free and complete. It is straighforward to check that $S_2(1) = (2, 1, 1/2)^T$ satisfies the previous set of equations and equation (3) gives

$$\lambda = \frac{1 - \frac{3}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{4}}{\frac{3}{2} - 2\frac{1}{3} + \frac{1}{4}} = \frac{15}{26},$$

and we can conclude that the unique risk neutral measure in the extended market is given by

$$Q = Q_{\frac{15}{26}} = \left(\frac{15}{26}, \frac{9}{26}, \frac{2}{26}\right)^T.$$

Note: The statement of the problem is misleading because it implies that if the market is complete then necessarily we must have a unique risk neutral measure. But it well may be that the set of risk neutral probability measures is empty. For instance, if $S_2(1) = (1, 2, 1)^T$ then the market is complete, because equation (2) is satisfied, but there are no risk neutral probability measures, because equations (4) and (5) are not satisfied.

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