STK-MAT3700

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Multiperiod Financial Markets

Problem A

Consider a market with $N = 1, K = 4$, $B(t) = (1+r)^t$, $r \geq 0$, $S(0) = 5$,

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}$$

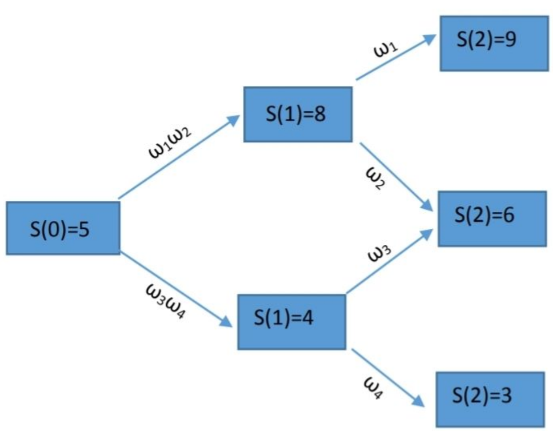
$$= 8\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + 4\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega),$$

$$S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2, \omega_3 \\ 3 & \text{if} \quad \omega = \omega_4 \end{cases}$$

$$= 9\mathbf{1}_{\{\omega_1\}}(\omega) + 6\mathbf{1}_{\{\omega_2,\omega_3\}}(\omega) + 3\mathbf{1}_{\{\omega_4\}}(\omega).$$

- Compute the filtration generated by S.
- Write down a generic H, V and G.
- Find RNPM.

Solution. Denote $\mathfrak{F}_0 = \{\varnothing, \Omega\}$, $\mathfrak{F}_1 = \{\varnothing, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$. Let $H = \{H(t) = (H_0(t), H_1(t))\}_{t=1,2}$ be a trading strategy, where



$\{H_i(1)\}_{i=0,1}$ is $\mathfrak{F}_0$ – measurable, $\{H_i(2)\}_{i=0,1}$ is $\mathfrak{F}_1$ – measurable. Then

$$V(0) = H_0(1,\omega) + 5H_1(1,\omega)$$

and

$$V(1,\omega) = \begin{cases} (1+r)H_0(1,\omega) + 8H_1(1,\omega) & \text{if} \quad \omega = \omega_1, \omega_2 \\ (1+r)H_0(1,\omega) + 4H_1(1,\omega) & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}$$

Since $\{H_i(1)\}_{i=0,1}$ is $\mathfrak{F}_0$ – measurable, then $H_0(1,\omega_i) = H_0(1)$ and $H_1(1,\omega_i) = H_1(1)$ for $i = 1, 2, 3, 4$.

$$\text{V}(2,\omega) = \begin{cases} (1+r)^2 H_0(2,\omega_1) + 9H_1(2,\omega_1) & \text{if} \quad \omega = \omega_1, \\ (1+r)^2 H_0(2,\omega_2) + 6H_1(2,\omega_2) & \text{if} \quad \omega = \omega_2, \\ (1+r)^2 H_0(2,\omega_3) + 6H_1(2,\omega_3) & \text{if} \quad \omega = \omega_3, \\ (1+r)^2 H_0(2,\omega_4) + 3H_1(2,\omega_4) & \text{if} \quad \omega = \omega_4. \end{cases}$$

$\{H_i(2)\}_{i=0,1}$ is $\mathfrak{F}_1$ – measurable, then

$$H_0(2,\omega_1) = H_0(2,\omega_1),$$

$$H_0(2,\omega_3) = H_0(2,\omega_4),$$

$$H_1(2,\omega_1) = H_1(2,\omega_1),$$

$$H_1(2,\omega_1) = H_1(2,\omega_1).$$

We compute $\triangle\ B(1) = r$, $\triangle\ B(2) = r(r+1)$,

$$\triangle\ S(1,\omega) = \begin{cases} 3 & \text{if} \quad \omega = \omega_1, \omega_2 \\ -1 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}$$

and

$$S(2,\omega) = \begin{cases} 1 & \text{if} \quad \omega = \omega_1 \\ -2 & \text{if} \quad \omega = \omega_2, \\ 2 & \text{if} \quad \omega = \omega_3, \\ -1 & \text{if} \quad \omega = \omega_4. \end{cases}$$

The discounted gains process $G^* = \{G^*(t)\}_{t=1,...,T}$ is defined by

$$G^*(t) = \sum_{n=1}^{N} \sum_{u=1}^{t} H_n(u)\, \Delta S_n^*(u), \quad t = 1, ..., T, \tag{1}$$

where $\Delta S_n^*(u) = S_n^*(u) - S_n^*(u-1)$. Then

$$G(1) = H_0(1) \vartriangle B(1) + H_1(1) \vartriangle S(1,\omega) = \begin{cases} rH_0 + 3H_1 & \text{if} \quad \omega = \omega_1, \omega_2 \\ rH_0 - H_1 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}$$

and

$$G(2) = G(1) + H_0(2) \vartriangle B(2) + H_1(2) \vartriangle S(2,\omega),$$

$$G(2) == \begin{cases} rH_0 + 3H_1 + r(r+1)H_0(2,\omega_1) + H_1(2,\omega_1) & \text{if} \quad \omega = \omega_1, \\ rH_0 + 3H_1 + r(r+1)H_0(2,\omega_1) - 2H_1(2,\omega_1) & \text{if} \quad \omega = \omega_2, \\ rH_0 - H_1 + r(r+1)H_0(2,\omega_1) + 2H_1(2,\omega_1) & \text{if} \quad \omega = \omega_3, \\ rH_0 - H_1 + r(r+1)H_0(2,\omega_1) - 1H_1(2,\omega_1) & \text{if} \quad \omega = \omega_4. \end{cases}$$

A trading strategy H is self-financing if and only if

$$V^*(t) = V^*(0) + G^*(t), \qquad t = 0, ..., T \tag{2}$$

For H to be self-financing we must have

$$V(1,\omega) = (1+r)H_0(1) + 8H_1(1) = (1+r)H_0(2,\omega) + 8H_1(2,\omega) \text{ if } \omega = \omega_1, \omega_2$$

$$V(1,\omega) = (1+r)H_0(1) + 4H_1(1) = (1+r)H_0(2,\omega) + 4H_1(2,\omega) \text{ if } \omega = \omega_3, \omega_4$$

By definition a risk neutral probability measure (martingale measure) is a probability measure Q such that

(1) $Q(\omega) > 0, \omega \in \Omega$.

(2) S_n^*, $n = 1, ..., N$ are martingales under Q, that is,

$$\mathbb{E}_Q\left[S_n^*(t+s) \middle| \mathcal{F}_t\right] = S_n^*(t), \qquad t, s \geq 0, n = 1, ..., N. \tag{3}$$

It suffices to check (3) for $s = 1$ and $t = 0, ..., T-1$, that is,

$$\mathbb{E}_Q\left[S_n^*(t+1) \middle| \mathcal{F}_t\right] = S_n^*(t).$$

If $B(t) = (1+r)^t$, then (3) is equivalent to

$$\mathbb{E}_Q\left[S_n(t+1) \middle| \mathcal{F}_t\right] = (1+r) S_n(t). \tag{4}$$

We will find $Q = (Q_1, Q_2, Q_3, Q_4)$ satisfying (4) for $t = 0, 1$.

At time $t = 0$:

$$S(0) = E_Q\left[\frac{S(1)}{1+r} | \mathfrak{F}_0\right] = E_Q\left[\frac{S(1)}{1+r}\right]$$

For given market we will have

$$8(Q_1 + Q_2) + 4(Q_1 + Q_2) = 5(1+r)$$

At time $t = 1$:

$$(1+r)S(1) = E_Q[S(2)|\mathfrak{F}_1] = E_Q[S(2)|\omega_1\omega_2]\mathbf{1}_{\{\omega_1,\omega_2\}} + E_Q[S(2)|\omega_3\omega_4]\mathbf{1}_{\{\omega_3,\omega_4\}} =$$

$$= \left(S(2,\omega_1)\frac{Q(\omega_1)}{Q(\omega_1,\omega_2)} + S(2,\omega_2)\frac{Q(\omega_2)}{Q(\omega_1,\omega_2)} \right) \mathbf{1}_{\{\omega_1,\omega_2\}} +$$

$$+ \left(S(2,\omega_3)\frac{Q(\omega_3)}{Q(\omega_3,\omega 4)} + (S(2,\omega_4)\frac{Q(\omega_4)}{Q(\omega_3,\omega_4)} \right) \mathbf{1}_{\{\omega_1,\omega_2\}} =$$

$$= \left(9\frac{Q(\omega_1)}{Q(\omega_1,\omega_2)} + 6\frac{Q(\omega_2)}{Q(\omega_1,\omega_2)} \right) \mathbf{1}_{\{\omega_1,\omega_2\}} +$$

$$+ \left(6\frac{Q(\omega_3)}{Q(\omega_3,\omega 4)} + 3\frac{Q(a_4)}{Q(\omega_3,\omega_4)} \right) \mathbf{1}_{\{\omega_1,\omega_2\}}.$$

Note that

$$S(1) = S(1,\omega)\mathbf{1}_{\{\omega_1,\omega_2\}} + S(1,\omega)\mathbf{1}_{\{\omega_3,\omega_4\}}$$

$$S(1) = 8 \cdot \mathbf{1}_{\{\omega_1,\omega_2\}} + 4 \cdot \mathbf{1}_{\{\omega_3,\omega_4\}}$$

Hence, we get

$$9Q_1 + 6Q_2 = 8(1+r)(Q_1+Q_2),$$

$$6Q_3 + 3Q_4 = 4(1+r)(Q_3+Q_4).$$

By definition RNPM we also have

$$Q_1 + Q_2 + Q_3 + Q_4 = 1.$$

As a result, we get the system with four equations

$$Q_1 + Q_2 + Q_3 + Q_4 = 1,$$

$$8(Q_1+Q_2) + 4(Q_1+Q_2) = 5(1+r),$$

$$9Q_1 + 6Q_2 = 8(1+r)(Q_1+Q_2),$$

$$6Q_3 + 3Q_4 = 4(1+r)(Q_3+Q_4).$$

We have four variable, four equations and we obtain

$$Q_1 = \frac{1+5r}{4} \cdot \frac{2+8r}{3}; \;\; Q_3 = \frac{3-5r}{4} \cdot \frac{1+4r}{3};$$

$$Q_2 = \frac{1+5r}{4} \cdot \frac{1-8r}{3}; \;\; Q_3 = \frac{3-5r}{4} \cdot \frac{2-4r}{3}.$$

Moreover $Q_i > 0$, for $i = 1, 2, 3, 4$. Therefore $0 \leq r < \frac{1}{8}$.

Problem B.

Consider the market from Problem A: $T = 2$, $K = 4$, $S(0) = 5$,

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}, \; S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2, \omega_3 \\ 3 & \text{if} \quad \omega = \omega_4 \end{cases}.$$

Define

- $X = (S(2) - 5)^+$. **European call option** with strike 5.

$$X = (\max(0, 9-5), \max(0, 6-5), \max(0, 6-5),$$
$$\max(0, 3-5))^T = (4, 1, 1, 0)^T.$$

- $Y = \left(\frac{1}{3}\sum_{t=0}^{2} S(t) - 5\right)^+$. **Asian call option** with strike 5.

$$Y_1 = \left(\frac{1}{3}\sum_{t=0}^{2} S(t,\omega_1) - 5\right)^+ = \max\left(0, \frac{1}{3}(5+8+9) - 5\right) = 7/3,$$

$$Y_2 = \left(\frac{1}{3}\sum_{t=0}^{2} S(t,\omega_2) - 5\right)^+ = \max\left(0, \frac{1}{3}(5+8+6) - 5\right) = 4/3,$$

$$Y_3 = \left(\frac{1}{3}\sum_{t=0}^{2} S(t,\omega_3) - 5\right)^+ = \max\left(0, \frac{1}{3}(5+4+6) - 5\right) = 0,$$

$$Y_4 = \left(\frac{1}{3}\sum_{t=0}^{2} S(t,\omega_4) - 5\right)^+ = \max\left(0, \frac{1}{3}(5+4+3) - 5\right) = 0,$$

which yields $Y = (7/3, 4/3, 0, 0)^T$.

Suppose $r = 0$. Consider $X = (S(2) - 5)^+$ and $Y = \left(\frac{1}{3}\sum_{t=0}^{2} S(t) - 5\right)^+$ or in vector notation $X = (4, 1, 1, 0)^T$ and $Y = (7/3, 4/3, 0, 0)^T$. Compute the price of X and Y for each t and a self-financing trading strategy generating X and Y. (Using different methods)

Solution. From the previous problem, we know that

$$Q_1 = \frac{1+5r}{4} \cdot \frac{2+8r}{3}; \;\; Q_3 = \frac{3-5r}{4} \cdot \frac{1+4r}{3};$$
$$Q_2 = \frac{1+5r}{4} \cdot \frac{1-8r}{3}; \;\; Q_3 = \frac{3-5r}{4} \cdot \frac{2-4r}{3}.$$

When substituting $r = 0$, we get $Q(\frac{1}{6}; \frac{1}{12}; \frac{1}{4}; \frac{1}{2})$ That is an unique martingale mesuare in this market.

We will use for European call option X the next method, which consist of two step:

- **Step 1.** We must know the value process $V = \{V(t)\}_{t=0,\ldots,T}$.

- **Step 2.** We solve

$$V(t) = H_0(t) + \sum_{n=1}^{N} H_n(t) S_n(t), \qquad t = 1, ..., T$$

taking into account that H must be predictable.

Note that, a contingent claim X is attainable (or marketable) if there exists a self-financing trading strategy such that $V(T) = X$. So,

$$V(2) = X,$$

and

$$V(0) = E_Q\left[\frac{X}{B(2)}\right] = \frac{1}{6} \cdot 4 + \frac{1}{12} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 0 = 1.$$

Recall $\mathfrak{F}_1 = \{\varnothing, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$. $V(1)$ and $V^*(1)$ are $\mathfrak{F}_1-$ measurable random variables.

$$V(1) = B(1) \cdot V^*(1) = B(1) E_Q[V^*(2)|\mathfrak{F}_1] = B(1) E_Q\left[\frac{X}{B(2)}|\mathfrak{F}_1\right].$$

In this case $B(1) = B(2) = 1$ and

$$V(1) = E_Q[X|\mathfrak{F}_1].$$

Moreover

$$E_Q[X|\mathfrak{F}_1](\omega) = E_Q[X|\{\omega_1, \omega_2\}]\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + E_Q[X|\{\omega_3, \omega_4\}]\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega),$$

where

$$E_Q[X|\{\omega_1, \omega_2\}] = \frac{E_Q[X \cdot \mathbf{1}_{\{\omega_1,\omega_2\}}]}{Q(\{\omega_1, \omega_2\})} = \frac{4 \cdot \frac{1}{6} + 1 \cdot \frac{1}{12} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2}}{\frac{1}{6} + \frac{1}{12}} = 3,$$

$$E_Q[X|\{\omega_3, \omega_4\}] = \frac{E_Q[X \cdot \mathbf{1}_{\{\omega_3,\omega_4\}}]}{Q(\{\omega_3, \omega_4\})} = \frac{0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{12} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2}}{\frac{1}{4} + \frac{1}{2}} = \frac{1}{3}.$$

Then for $t = 2$:

$$\begin{aligned} V(2, \omega_1) &= 4 = H_0(2, \omega_1) \cdot 1 + H_1(2, \omega_1) \cdot 9, \\ V(2, \omega_2) &= 1 = H_0(2, \omega_2) \cdot 1 + H_1(2, \omega_2) \cdot 6, \\ V(2, \omega_3) &= 1 = H_0(2, \omega_3) \cdot 1 + H_1(2, \omega_3) \cdot 6, \\ V(2, \omega_3) &= 0 = H_0(2, \omega_3) \cdot 1 + H_1(2, \omega_3) \cdot 3. \end{aligned}$$

$\{H_i(2)\}_{i=0,1}$ is $\mathfrak{F}_1-$measurable, then $H_0(2, \omega_1) = H_0(2, \omega_1)$, $H_0(2, \omega_3) = H_0(2, \omega_4)$, $H_1(2, \omega_1) = H_1(2, \omega_1)$, $H_1(2, \omega_1) = H_1(2, \omega_1)$.

We get

$$H_0(2, \omega) = \begin{cases} 5 & \text{if} \quad \omega = \omega_1, \omega_2 \\ -1 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}$$

$$H_1(2,\omega) = \begin{cases} 1 & \text{if } \omega = \omega_1, \omega_2 \\ -\frac{1}{3} & \text{if } \omega = \omega_3, \omega_4 \end{cases}$$

At time $t = 1$:

$$V(1,\omega) = 3 = H_0(1,\omega)\cdot 1 + H_1(1,\omega)\cdot 8, \omega = \omega_1, \omega_2;$$
$$V(1,\omega) = \frac{1}{3} = H_0(1,\omega)\cdot 1 + H_1(1,\omega)\cdot 4, \omega = \omega_3, \omega_4$$

with

$$H_0(1,\omega_1) = H_0(1,\omega_2) = H_0(1,\omega_3) = H_0(1,\omega_4);$$
$$H_1(1,\omega_1) = H_1(1,\omega_2) = H_1(1,\omega_3) = H_1(1,\omega_4).$$

Then

$$H_0(1,\omega) = -\frac{7}{3}, \;\; H_1(1,\omega) = \frac{2}{3}, \omega \in \Omega.$$

For Asian call option $Y = (\frac{7}{3}, \frac{4}{3}, 0, 0)$ we will use method which relies on the fact that the self-financing condition

$$V^*(0) + G^*(t) = V^*(t),$$

is equivalent to

$$V^*(t-1) + \sum_{n=1}^{N} H_n(t)\,\Delta S_n^*(t) = V^*(t).$$

We can use this system of equations, together with the predictability condition on $H(t) = (H_1(t), ..., H_N(t))$, to find $V^*(t-1)$ and $H(t)$.

Then, we can find

$$H_0(t) = V^*(t-1) - \sum_{n=1}^{N} H_n(t)\, S_n^*(t-1),$$
$$V(t-1) = B(t-1)\, V^*(t-1).$$

We begin with $V^*(T) = X/B(T)$ and work backwards in time.

Recall that $\Delta S^*(2) = (1, -2, 2, -1)$ and $\Delta S^*(1) = (3, 3, -1, -1)$ we have

$$V^*(1,\omega) + H_1(2,\omega)\cdot 1 = \frac{7}{3},$$
$$V^*(1,\omega) + H_1(2,\omega)\cdot(-2) = \frac{4}{3},$$
$$V^*(1,\omega) + H_1(2,\omega)\cdot(2) = 0,$$
$$V^*(1,\omega) + H_1(2,\omega)\cdot(-1) = 0.$$

Then $V^*(1,\omega) = 2$, $H_1(2,\omega) = \frac{1}{3}$ for $\omega = \omega_1, \omega_2$ and $V^*(1,\omega) = 0$, $H_1(2,\omega) = 0$ for $\omega = \omega_3, \omega_4$.

For $t = 0$:

$$V^*(0,\omega) + H_1(2,\omega) \cdot 3 = 2 = V^*(1,\omega), \text{for } \omega = \omega_1, \omega_2;$$

$$V^*(0,\omega) + H_1(2,\omega) \cdot (-1) = 0 = V^*(1,\omega), \text{for } \omega = \omega_3, \omega_4.$$

Then $V^*(0,\omega) = \frac{1}{2}$ and $H(1,\omega) = \frac{1}{2}$ for all ω.
To compute H_0 we use

$$H_0(1) = V^*(0) - H_1(1)S(0) = \frac{1}{2} - \frac{1}{2} \cdot 5 = -2;$$

$$H_0\,(2) = V^*(1) - H_1(2)S(1) = \begin{cases} 2 - \frac{1}{3} \cdot 8 = -\frac{2}{3} & \text{if } \omega = \omega_1, \omega_2; \\ 0 - 0 \cdot 4 = 0 & \text{if } \omega = \omega_3, \omega_4. \end{cases}$$

Note, that $V^*(0) = \frac{1}{2}$ is the same value obtained using the risk pricing approach, i.e.,

$$\frac{1}{2} = E_Q\left[\frac{Y}{B(2)}\right].$$

References

[1] S. R. Pliska. Introduction to Mathematical Finance. Discrete Time Models. Blackwell Publishing. (1997)