Arbitrage opportunity.Risk neutral probability measures.Valuation of contingent claims

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Arbitrage opportunity

Arbitrage opportunity

An ***arbitrage opportunity*** (**AO**) is a trading strategy satisfying:

a) $V(0) = 0.$

b) $V(1, \omega) \geq 0, \quad \omega \in \Omega.$

c) $\mathbb{E}[V(1)] > 0.$

1. c) can be changed by
   - c') $\exists\, \omega \in \Omega$ such that $V(1, \omega) > 0.$
2. a), b) c) $\Longleftrightarrow V^*(0) = 0, V^*(1) \geq 0$, and $\mathbb{E}[V^*(1)] > 0.$
3. An **AO** is a trading strategy
   - with zero initial investment,
   - without the possibility of bearing a loss
   - with a strictly positive profit for at least one of the possible states of the economy.

Arbitrage opportunity

1. $\exists$ **DTS** $\Longrightarrow$ $\exists$ **AO.**
2. $\exists$ **AO** $\exists$ **DTS.**

Proof.

1. We know that

   $\exists$ of **DTS** $\Longleftrightarrow$ $\exists$ of H such that $V(0) = 0$ and $V(1, \omega) > 0, \omega \in \Omega$.

   But, if $V(1, \omega) > 0, \omega \in \Omega$ then

$$\mathbb{E}\left[V(1)\right] = \sum_{\omega \in \Omega} V(1, \omega) P(\omega) > 0.$$

2. The following example provides a counterexample.

□

Arbitrage opportunity

Example

- Take $K = 2, N = 1, r = 0, B(0) = 1, B(1) = 1, S(0) = S^*(0) = 10$ and

$$S(1,\omega) = S^*(1,\omega) = \begin{cases} 12 & \text{if} \quad \omega = \omega_1 \\ 10 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- Consider the trading strategy $H = (H_0, H_1)^T = (-10, 1)^T$, then $V(0) = H_0 B(0) + H_1 S(0) = -10 + 10 = 0$, and

$$V(1) = H_0 B(1) + H_1 S(1) = \begin{cases} -10 + 12 = 2 & \text{if} \quad \omega = \omega_1 \\ -10 + 10 = 0 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- Hence, H is an arbitrage opportunity.

Arbitrage opportunity

Example 1

- We know that the model does not contain **DTS** if and only if $\exists$ **LPM**.
- A **LPM** $\pi = (\pi_1, \pi_2)^T$ must satisfy $\pi \geq 0$ and

$$10 = S^* (0) = \mathbb{E}_\pi [S^* (1)] = 12\pi_1 + 10\pi_2.$$

- Hence, $\pi = (0, 1)^T$ is a **LPM** and we can conclude.

Arbitrage opportunity

H is an **AO** $\Longleftrightarrow G^*(\omega) \geq 0, \omega \in \Omega$ and $\mathbb{E}[G^*] > 0$.

Risk Neutral Probability Measures

Risk neutral probability measures

- Recall that $\exists$ **LPM** $\implies \nexists$ **DTS**, but there may be **AO**.
- In order to rule out **AO** we need to narrow the concept of **LPM**.
- The idea is to require that a **LPM** must assign a strictly positive probability to each state of the economy.
- Equivalently, a **LPM**, say π, must be equivalent to P, that is,

$$P(\omega) > 0 \iff \pi(\omega) > 0, \qquad \omega \in \Omega.$$

A probability measure Q is called a ***risk neutral probability measure*** (**RNPM**) if

1. $Q(\omega) > 0, \qquad \omega \in \Omega.$
2. $\mathbb{E}_Q[\Delta S_n^*] = 0, \qquad n = 1, ..., N.$

Given a financial market model, we will denote by $\mathbb{M}$ the set of all **RNPM.**

Risk neutral probability measures

- Observe that

$$0 = \mathbb{E}_Q [\Delta S_n^*] = \mathbb{E}_Q [S_n^* (1) - S_n^* (0)] = \mathbb{E}_Q [S_n^* (1)] - S_n^* (0) \, .$$

- That is, $\mathbb{E}_Q [S_n^* (1)] = S_n^* (0) \, .$
- Therefore, Q is a **LPM**.

[First Fundamental Theorem of Asset Pricing (**FFTAP**)] $\nexists$ **AO** $\Longleftrightarrow \exists$ **RNPM** (that is, $\mathbb{M} \neq \emptyset$).

Risk neutral probability measures

Example (∃! **RNPM**)

- Take $K = 2, N = 1, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S(0) = 5$, and

$$S^*(1, \omega) = \begin{cases} 6 & \text{if} \quad \omega = \omega_1 \\ 4 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- We are seeking a probability measure $Q = (Q_1, Q_2)^T$ such that

$$\mathbb{E}_Q[\Delta S^*] = 0 \Longleftrightarrow \mathbb{E}_Q[S^*(1)] = S^*(0) = 5$$
$$\Longleftrightarrow \begin{cases} 6Q_1 + 4Q_2 & = 5 \\ Q_1 + Q_2 & = 1 \end{cases}.$$

- ∃! solution to the previous equation given by $Q = (1/2, 1/2)$.
- Therefore, Q is a **RNPM** and the market is arbitrage free by the **FFTAP**.

Risk neutral probability measures

Example ($\exists\infty$ **RNPM**)

- Take $K = 3, N = 1, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S(0) = 5$, and

$$S^*(1,\omega) = \begin{cases} 6 & \text{if} \quad \omega = \omega_1 \\ 4 & \text{if} \quad \omega = \omega_2 \\ 3 & \text{if} \quad \omega = \omega_3 \end{cases}.$$

- For $Q = (Q_1, Q_3, Q_3)^T$ to be a **RNPM**, Q must satisfy

$$\mathbb{E}_Q[\Delta S^*] = 0 \iff \mathbb{E}_Q[S^*(1)] = S^*(0) = 5$$
$$\iff \begin{cases} 6Q_1 + 4Q_2 + 3Q_3 & = 5 \\ Q_1 + Q_2 + Q_3 & = 1 \end{cases}.$$

- We have 2 equations and 3 unknowns (underdetermined system).

Risk neutral probability measures

Example 3 ($\exists\infty$ RNPM)

- In addition, we also have the restrictions $Q_i > 0, i = 1, 2, 3.$
- Solving the equations, taking into account the constraints, we obtain a family of **RNPM**

$$Q_\lambda = (\lambda, 2 - 3\lambda, -1 + 2\lambda)^T, \qquad \lambda \in (1/2, 2/3).$$

- Now there are infinitely many **RNPM** (one for each λ) and, again, the market is arbitrage free by the **FFTAP**.

Valuation of Contingent Claims

Valuation of contingent claims

A ***contingent claim*** is a random variable X representing a payoff at time $t = 1$.

- Think of a contingent claim as any financial contract with some payoff at time $t = 1$ (options for instance).

A contingent claim is said to be ***attainable*** (or ***marketable***) if there exists a trading strategy H, called the ***replicating/hedging*** portfolio, such that $V(1) = X$. We say that H ***generates/replicates/hedge*** X.

Valuation of contingent claims

- Suppose that the contingent claim X is attainable, i.e., $V(1) = X$.
- Suppose also that it can be bought in the market (at time 0) for the price $p(X)$.
- Then, using the no arbitrage pricing principle:
  - If $p(X) > V(0)$:
    - At $t = 0$: Sell the claim (receive $p(X)$), implement X (that is, $V(1)$ at cost $V(0)$) and invest $p(X) - V(0)$ risk free.
    - At $t = 1 : -X + V(1) + (p(X) - V(0))(1 + r) > 0$.
  - If $p(X) < V(0)$:
    - At $t = 0$: Buy the claim (pay $p(X)$), implement $-X$ (that is, $-V(1)$ receiving $V(0)$) and invest $V(0) - p(X)$ risk free.
    - At $t = 1 : X - V(1) + (V(0) - p(X))(1 + r) > 0$.
- Does this mean that $p(X) = V(0)$ is the correct price for X? Not necessarily.
- Suppose that $\exists \widehat{H}$ such that $\widehat{V}(1) = X$ and $\widehat{V}(0) \neq V(0)$.
- This second strategy could be used to generate an arbitrage if $p(X) = V(0)$.

Valuation of contingent claims

- In order to rule out this possibility we need to assume that **LOP** holds.
- We have just proved the following result.

If **LOP** holds, then the price $p(X)$ ($t = 0$ value) of an attainable contingent claim X is given by

$$p(X) = V(0) = H_0 B(0) + \sum_{n=1}^{N} H_n S_n(0), \quad (1)$$

where H is any trading strategy that generates X.

- Recall that $\nexists$ **AO** $\implies$ $\nexists$ **DTS** $\implies$ **LOP** holds.

Valuation of contingent claims

Assume $\nexists$ **AO**. Then, the price $p(X)$ of any attainable contingent claim X is given by

$$p(X) = \mathbb{E}_Q\left[\frac{X}{B(1)}\right], \tag{2}$$

where Q is any **RNPM** in $\mathbb{M}$.

Valuation of contingent claims

Example

- Take $K = 2, N = 1, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S(0) = 5$,

$$S^*(1,\omega) = \begin{cases} 6 & \text{if} \quad \omega = \omega_1 \\ 4 & \text{if} \quad \omega = \omega_2 \end{cases},$$

  and

$$S(1,\omega) = \begin{cases} 6\frac{10}{9} = \frac{20}{3} & \text{if} \quad \omega = \omega_1 \\ 4\frac{10}{9} = \frac{40}{9} & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- Recall that in this market there is only one RNPM $Q = (1/2, 1/2)^T$.
- Let X be the contingent claim defined by

$$X(\omega) = \begin{cases} 7 & \text{if} \quad \omega = \omega_1 \\ 2 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

Valuation of contingent claims

- Suppose that X is attainable, then the price of X is given by

$$p(X) = \mathbb{E}_Q\left[\frac{X}{B(1)}\right] = \frac{7}{\frac{10}{9}}\frac{1}{2} + \frac{2}{\frac{10}{9}}\frac{1}{2} = \frac{81}{20}.$$

- Let's prove that X is indeed attainable. We want to find $H = (H_0, H_1)^T$that generates X, that is,

$$\frac{X}{B(1)} = V^*(1) = V^*(0) + G^* = V^*(0) + H_1 \Delta S^*.$$

- Since $V^*(0) = V(0) = p(X) = \frac{81}{20}$ and

$$\Delta S^* = \begin{cases} 6-5=1 & \text{if} \quad \omega = \omega_1 \\ 4-5=-1 & \text{if} \quad \omega = \omega_2 \end{cases},$$

Valuation of contingent claims

we get the following equations

$$\frac{7}{\frac{10}{9}} = \frac{81}{20} + H_1,$$
$$\frac{2}{\frac{10}{9}} = \frac{81}{20} - H_1.$$

- These two equations are compatible and $H_1 = \frac{9}{4}$.
- To determine H_0 we can use

$$\frac{81}{20} = V(0) = H_0 B(0) + H_1 S(0) = H_0 + \frac{9}{4}5,$$

which yields $H_0 = -\frac{36}{5}$.

Valuation of contingent claims

- The interpretation is as follows:
  - At $t = 0$:
    - You sell the claim and get $V(0) = \frac{81}{20}$.
    - You hedge the claim by borrowing $-H_0 = \frac{36}{5}$ at interest $\frac{1}{9}$, using $V(0) - H_0 = \frac{81}{20} + \frac{36}{5} = \frac{45}{4}$ to buy $H_1 = \frac{V(0)-H_0}{S(0)} = \frac{\frac{45}{4}}{5} = \frac{9}{4}$ shares of the stock.
  - At $t = 1$:
    - Pay $-H_0 B(1) = \frac{36}{5}\frac{10}{9} = 8$ to the bank to close the loan.
    - The value of the portfolio is

$$V(1) = H_0 B(1) + H_1 S(1) = -8 + \frac{9}{4} S(1)$$
$$= \begin{cases} -8 + \frac{9}{4}\frac{20}{3} = 7 & \text{if} \quad \omega = \omega_1 \\ -8 + \frac{9}{4}\frac{40}{9} = 2 & \text{if} \quad \omega = \omega_2 \end{cases},$$

and you can pay the contingent claim sold.

Valuation of contingent claims

- Now, suppose that we add a third state ω_3 in the economy and $S^*(1,\omega_3) = 3$ and $S(1,\omega_3) = \frac{10}{3}$.
- This is the same extension as in Example 3, so we know $\exists\infty$ **RNPM.**
- Consider an arbitrary contingent claim X in this market, that is,

$$X(\omega) = \begin{cases} X_1 & \text{if} \quad \omega = \omega_1 \\ X_2 & \text{if} \quad \omega = \omega_2 \\ X_3 & \text{if} \quad \omega = \omega_3 \end{cases} = (X_1, X_2, X_3)^T.$$

- X is attainable if there exists $H = (H_0, H_1)^T$ such that

$$X = V(1) = H_0 B(0) + H_1 S(1).$$

Valuation of contingent claims

- The previous vector equation boils down to the following overdetermined linear system

$$\begin{cases} X_1 &= \frac{10}{9}H_0 + \frac{20}{3}H_1 \\ X_2 &= \frac{10}{9}H_0 + \frac{40}{9}H_1 \\ X_3 &= \frac{10}{9}H_0 + \frac{10}{3}H_1 \end{cases} .$$

- From the first equation we obtain $\frac{10}{9}H_0 = X_1 - \frac{20}{3}H_1$ and substituting this expression for $\frac{10}{9}H_0$ in the second and third equations we get

$$\begin{cases} X_2 &= X_1 - \frac{20}{3}H_1 + \frac{40}{9}H_1 = X_1 - \frac{20}{9}H_1 \\ X_3 &= X_1 - \frac{20}{3}H_1 + \frac{10}{3}H_1 = X_1 - \frac{10}{3}H_1 \end{cases} .$$

Valuation of contingent claims

- The first equation in the previous system gives

$$H_1 = \frac{9}{20}\left(X_2 - X_1\right),$$

and the second equation gives

$$H_1 = \frac{3}{10}\left(X_3 - X_1\right).$$

- Therefore, equating the previous expressions for H_1, we obtain.

$$\frac{9}{20}\left(X_2 - X_1\right) = \frac{3}{10}\left(X_3 - X_1\right) \iff X_1 - 3X_2 + 2X_3 = 0. \qquad (3)$$

- We can conclude that a contingent claim $X = (X_1, X_2, X_3)^T$ in this market is attainable if and only if X satisfies equation (3).

Valuation of contingent claims

Example

- In a general single period model consider the so called ***counting claim*** X defined by
$$X(\omega) = \begin{cases} 1 & \text{if} \quad \omega = \widehat{\omega} \\ 0 & \text{if} \quad \omega \neq \widehat{\omega} \end{cases},$$
for some $\widehat{\omega} \in \Omega$.
- Assuming that X is attainable we have that
$$p(X) = \mathbb{E}_Q\left[\frac{X}{B(1)}\right] = \sum_{\omega \in \Omega} \frac{X(\omega)}{B(1)} Q(\omega) = \frac{Q(\widehat{\omega})}{B(1)} =: p(\widehat{\omega}).$$
- $p(\widehat{\omega})$ is called the state price for state $\widehat{\omega}$.
- The price of any contingent claim X can be obtained as the weighted sum of its payoff where the weights are the state prices, i.e., $p(X) = \sum_{\omega \in \Omega} X(\omega) p(\omega)$.

Complete and Incomplete Markets

Complete and Incomplete Markets

A financial market model is ***complete*** if every contingent claim X is attainable.
Otherwise, we say that the market model is ***incomplete***.

- So far, in order to use the risk neutral pricing principle to find the price of a contingent claim X, we need to ensure that the contingent claim is attainable.
- Therefore, it is important to find useful criteria to decide if a claim is attainable and, more generally, if the market is complete.

Complete and Incomplete Markets

The market is complete $\iff \operatorname{rank}(S(1,\Omega)) = K$.

Proof.

- Let $H = (H_0, H_1, ..., H_n)^T \in \mathbb{R}^{N+1}$ be a trading strategy and $X = (X_1, ..., X_K)^T \in \mathbb{R}^K$ a contingent claim.
- The market is complete $\iff S(1,\Omega)\, H = X$ has a solution in H for every $X \iff$ Linear span of the columns of $S(1,\Omega)$ is $\mathbb{R}^K \iff \dim(\operatorname{col}(S(1,\Omega))) = K$.
- But note that

$$\operatorname{rank}(S(1,\Omega)) = \dim(\operatorname{col}(S(1,\Omega))) = \dim(\operatorname{row}(S(1,\Omega))).$$

- That is, if $S(1,\Omega)$ has K linear independent columns or rows.

□

Complete and Incomplete Markets

Example (Continuation of Example 2)

- Take $K = 2, N = 1, r = \frac{1}{9}, B(0) = 1, B(1) = \frac{10}{9}, S_1(0) = 5$, and

$$S_1(1,\omega) = \begin{cases} \frac{20}{3} & \text{if} \quad \omega = \omega_1 \\ \frac{40}{9} & \text{if} \quad \omega = \omega_2 \end{cases} .$$

- Recall that this market is arbitrage free and it has a unique **RNPM** given by $Q = \left(\frac{1}{2}, \frac{1}{2}\right)^T$.
- Moreover,

$$S(1,\Omega) = \begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ \frac{10}{9} & \frac{40}{9} \end{pmatrix} \sim_{R_2 \rightsquigarrow R_2 - R_1} \begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ 0 & -\frac{20}{9} \end{pmatrix},$$

and we can conclude that $\text{rank}(S(1,\Omega)) = 2 = K$ and the market is complete.

Complete and Incomplete Markets

Example 6

- In the same market we add a second asset with $S_2(0) = 54$ and

$$S_2(1,\omega) = \begin{cases} 70 & \text{if} \quad \omega = \omega_1 \\ 50 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- We have that

$$\mathbb{E}_Q\left[S_2^*(1)\right] = \frac{70}{\frac{10}{9}}\frac{1}{2} + \frac{50}{\frac{10}{9}}\frac{1}{2} = 54 = S_2^*(0),$$

  and, therefore, Q is also a **RNPM** in the extended market.

- Moreover,

$$S(1,\Omega) = \begin{pmatrix} \frac{10}{9} & \frac{20}{3} & 70 \\ \frac{10}{9} & \frac{40}{9} & 50 \end{pmatrix} \sim_{R_2 \rightsquigarrow R_2 - R_1} \begin{pmatrix} \frac{10}{9} & \frac{20}{3} & 70 \\ 0 & \frac{-20}{9} & -20 \end{pmatrix},$$

  so the $\mathrm{rank}\left(S(1,\Omega)\right) = \dim\left(\mathrm{row}\left(S(1,\Omega)\right)\right) = 2 = K$ and the market is also complete.

Complete and Incomplete Markets

Example (Continuation of Example 3)

- Take $K=3, N=1, r=\frac{1}{9}, B(0)=1, B(1)=\frac{10}{9}, S(0)=5$, and

$$S^*(1,\omega)=\begin{cases} 6 & \text{if} \quad \omega=\omega_1 \\ 4 & \text{if} \quad \omega=\omega_2 \\ 3 & \text{if} \quad \omega=\omega_3 \end{cases}.$$

- In this market we have a family of **RNPM**

$$Q_\lambda=(\lambda, 2-3\lambda, 2\lambda-1)^T, \qquad \lambda\in(1/2,2/3).$$

- Moreover, the market is incomplete since

$$S(1,\Omega)=\begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ \frac{10}{9} & \frac{40}{9} \\ \frac{10}{9} & \frac{30}{9} \end{pmatrix} \sim_{\substack{R_2\rightsquigarrow R_2-R_1 \\ R_3\rightsquigarrow R_3-R_1}} \begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ 0 & -\frac{20}{9} \\ 0 & -\frac{30}{9} \end{pmatrix},$$

and the $\operatorname{rank}(S(1,\Omega))=\dim(\operatorname{col}(S(1,\Omega)))=2\neq K=3.$

Complete and Incomplete Markets

Example 7

- For any contingent claim X and any **RNPM** Q_λ we have

$$\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] = \lambda\frac{9}{10}X_1 + (2-3\lambda)\frac{9}{10}X_2 + (2\lambda-1)\frac{9}{10}X_3$$
$$= \frac{9}{10}\lambda(X_1 - 3X_2 + 2X_3) + \frac{9}{10}(2X_2 - X_1).$$

- If X is attainable this value must be the same for all $\lambda \in \left(\frac{1}{2}, \frac{2}{3}\right)$ because it must coincide with $V(0)$, which does not depend on Q_λ.
- Note that this happens if and only if

$$X_1 - 3X_2 - 2X_3 = 0.$$

- Recall (see Example 4) that this condition also characterizes the attainable contingent claims in this market.
- This is a general principle.

Complete and Incomplete Markets

Suppose that $\mathbb{M} \neq \emptyset$. Then,
A contingent claim X is attainable $\Longleftrightarrow \mathbb{E}_Q\left[\frac{X}{B(1)}\right]$ is constant with respect to $Q \in \mathbb{M}$.

Proof.

Smartboard. □

[Second Fundamental Theorem of Asset Pricing (**SFTAP**)] Suppose that $\mathbb{M} \neq \emptyset$. Then,
The market model is complete $\Longleftrightarrow \mathbb{M} = \{Q\}$, that is, $\exists!$ **RNPM**.

Proof.

Smartboard. □

Complete and Incomplete Markets

- Summarizing, we know how to price all attainable claims in a single period financial market.
- But, what about non-attainable claims in an incomplete model?
- We need some new concepts.

Let X be a non-attainable contingent claim. Then,

1. The **upper hedging price** of X, denoted by $V_{+}(X)$, is defined as

$$V_{+}(X) := \inf\left\{\mathbb{E}_Q\left[\frac{Y}{B(1)}\right] : Y \geq X, \quad Y \text{ is attainable}\right\}.$$

2. The **lower hedging price** of X, denoted by $V_{-}(X)$, is defined as

$$V_{-}(X) := \sup\left\{\mathbb{E}_Q\left[\frac{Y}{B(1)}\right] : Y \leq X, \quad Y \text{ is attainable}\right\}.$$

Complete and Incomplete Markets

[An analogous remark apply to $V_{-}(X)$]

1. $V_{+}(X)$ is well defined and it is finite.
   - For any $\lambda > 0$, $\lambda B(1)$ is an attainable claim and if λ is large enough ($\lambda = \max_k \left\{ \frac{X_k}{B(1)} \right\}$) we have $\lambda B(1) \geq X$.
   - Hence, $V_{+}(X) \leq \mathbb{E}_Q \left[\frac{\lambda B(1)}{B(1)} \right] = \lambda < +\infty$.
   - We also have that

$$
\begin{aligned}
V_{+}(X) &:= \inf_{Y \geq X,\, Y \text{ is attainable}} \left\{ \mathbb{E}_Q \left[\frac{Y}{B(1)} \right] \right\} \\
&\geq \inf_{Y \geq X,\, Y \text{ is attainable}} \left\{ \mathbb{E}_Q \left[\frac{X}{B(1)} \right] \right\} \\
&= \mathbb{E}_Q \left[\frac{X}{B(1)} \right] \geq \min_k \left\{ \frac{X_k}{B(1)} \right\} > -\infty.
\end{aligned}
$$

   - Since this inequality holds for all $Q \in \mathbb{M}$, it follows that

$$
V_{+}(X) \geq \sup \left\{ \mathbb{E}_Q \left[\frac{X}{B(1)} \right] : Q \in \mathbb{M} \right\}.
$$

Complete and Incomplete Markets

Remark

2
- $V_+(X)$ provides a good upper bound on the fair price of X in the sense that is the price of the cheapest portfolio that can be used to hedge a short position on X.
- If you sell the contingent claim X for more than $V_+(X)$ you can make a risk-less profit.

- Therefore, the fair price of X must lie in the interval $[V_-(X), V_+(X)]$.
- So we are interested in computing $V_+(X)$ as well as any attainable contingent claim $Y \geq X$ such that $V_+(X) = \mathbb{E}_Q\left[\frac{Y}{B(1)}\right]$.

Complete and Incomplete Markets

If $\mathbb{M} \neq \emptyset$, then for any contingent claim X one has

$$V_+(X) = \sup\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right] : Q \in \mathbb{M}\right\}$$

and

$$V_-(X) = \inf\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right] : Q \in \mathbb{M}\right\}.$$

Note that if X is attainable

$$V_+(X) = V_-(X) = \mathbb{E}_Q\left[\frac{X}{B(1)}\right],$$

for any $Q \in \mathbb{M}$.

Complete and Incomplete Markets

Example (Continuation Examples 3 and 7)

- Consider the market with $B(0) = 1, S(0) = 5$ and payoff matrix

$$S(1, \Omega) = \begin{pmatrix} \frac{10}{9} & \frac{20}{3} \\ \frac{10}{9} & \frac{40}{9} \\ \frac{10}{9} & \frac{30}{9} \end{pmatrix}.$$

- In this market we have a family of **RNPM**

$$\mathbb{M} = \left\{ Q_\lambda = (\lambda, 2 - 3\lambda, 2\lambda - 1)^T, \lambda \in \left(\frac{1}{2}, \frac{2}{3} \right) \right\},$$

and $X = (X_1, X_2, X_3)^T$ is attainable if and only if

$$X_1 - 3X_2 - 2X_3 = 0.$$

- Take $X = (30, 20, 10)^T$, which is not attainable because $30 - 3 \times 20 - 2 \times 10 \neq -50$.

Complete and Incomplete Markets

Example 8

- Then, we compute

$$\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] = \lambda\frac{9}{10}30 + (2-3\lambda)\frac{9}{10}20 + (2\lambda-1)\frac{9}{10}10$$
$$= 27 - 9\lambda.$$

- This gives

$$V_+(X) = \sup_{Q\in\mathbb{M}}\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right]\right\} = \sup_{\lambda\in\left(\frac{1}{2},\frac{2}{3}\right)}\{27-9\lambda\}$$
$$= 27 - 9\frac{1}{2} = 22.5,$$
$$V_-(X) = \inf_{Q\in\mathbb{M}}\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right]\right\} = \inf_{\lambda\in\left(\frac{1}{2},\frac{2}{3}\right)}\{27-9\lambda\}$$
$$= 27 - 9\frac{2}{3} = 21.$$

Complete and Incomplete Markets

Example 8

- Any price of X in the interval $[21, 22.5]$ is arbitrage free.
- By solving appropriate **LP** problems one can find attainable claims corresponding to the upper and lower hedging prices $V_+(X)$ and $V_-(X)$.
- In fact, one can check that
  - $Y = (30, 20, 15)^T \geq (30, 20, 10)^T = X$ gives
  $$V_+(X) = \mathbb{E}_{Q_\lambda}\left[\frac{Y}{B(1)}\right], \qquad \lambda \in \left(\frac{1}{2}, \frac{2}{3}\right).$$
  - $Y = \left(30, \frac{50}{3}, 10\right)^T \leq (30, 20, 10)^T = X$ gives
  $$V_-(X) = \mathbb{E}_{Q_\lambda}\left[\frac{Y}{B(1)}\right], \qquad \lambda \in \left(\frac{1}{2}, \frac{2}{3}\right).$$

Thank you!