Optimal Portfolio Problem. Risk Neutral Computational Approach.

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Oslo 2022.10.16

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Contents

Optimal Portfolio Problem (OPP)

Introduction

- The goal of an investor is transforming wealth invested at time $t = 0$ into wealth at time $t = 1$.
- The goal in this section will be to choose the “best” trading strategy.
- To be able to talk about “best” we need a measure of performance.
- We need to introduce the concept of utility function.

Utility functions

Utility function

A functions $U : \mathbb{R} \times \Omega \to \mathbb{R}$ is called a **utility function** if for each $\omega \in \Omega$ fixed the function $u \mapsto U(u, \omega)$ is

1. differentiable,
2. concave,
3. strictly increasing $\left(\frac{\partial}{\partial u} U(u, \omega) > 0,\ \omega \in \Omega\right)$.

- For many applications it suffices for U to depend only on the wealth argument u and not on $\omega \in \Omega$.

Utility functions

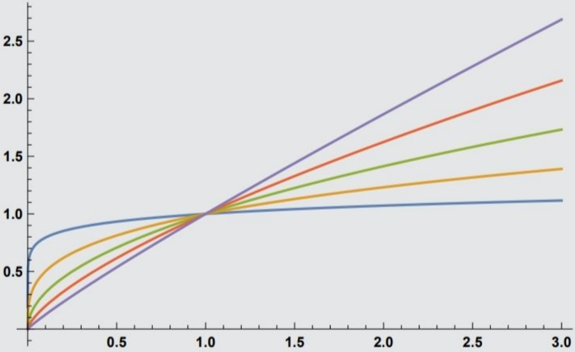
- If $V(1)$ is the portfolio value at $t=1$, then $U(V(1))$ represents the utility of the wealth $V(1)$. $(U(V(1,\omega),\omega),\ \omega\in\Omega)$.
- U being increasing: More wealth $\Longrightarrow$ More utility.
- U being concave: More wealth $\Longrightarrow$ Less marginal utility (saturation effect)
- Our measure of performance will be the expected utility of the final wealth, that is,

$$\mathbb{E}\left[U(V(1))\right]=\sum_{\omega\in\Omega}U(V(1,\omega),\omega)P(\omega).$$

Utility functions

Example (Utility functions)

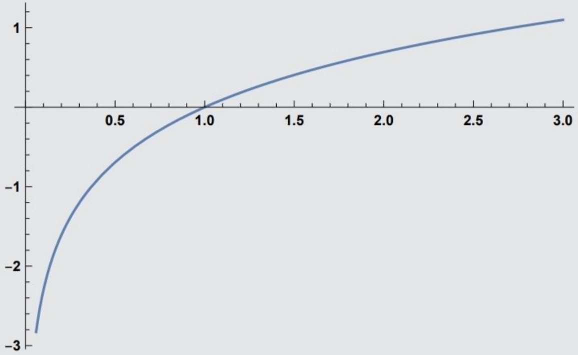
- $U_1(u) = u^{\gamma}, \quad u > 0, \gamma \in (0,1).$



Utility functions

Example

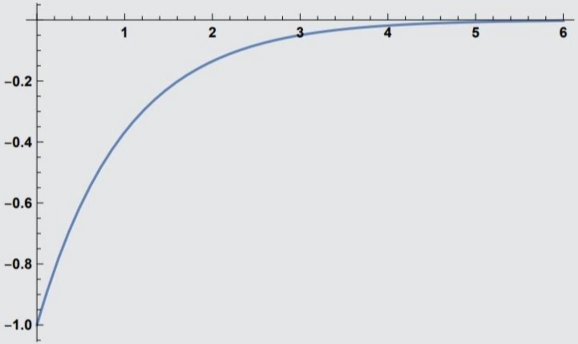
- $U_2(u) = \log(u), \quad u > 0.$



Utility functions

Example

- $U_3(u) = -e^{-u}, \quad u > 0.$



Optimization problem

- Given an initial wealth $v \in \mathbb{R}$, we can consider the set of strategies $H \in \mathbb{R}^{N+1}$ such that

$$v = V(0) = H_0 B(0) + \sum_{n=1}^{N} H_n S_n(0),$$

which impose some constraints on H, and try to maximize the expected utility of the terminal wealth.

- That is,

Optimal Portfolio Problem $(\mathbf{OPP}(v, U))$

$$\left.\begin{array}{cc} \max & \mathbb{E}\left[U\left(V\left(1\right)\right)\right] \\ \text{subject to} & V(0) = v \in \mathbb{R}, \\ & H \in \mathbb{R}^{N+1}, \end{array}\right\} \qquad (1)$$

Optimization problem

- Taking into account that $V(1) = B(1)\,V^*(1)$ and

$$V^*(1) = V^*(0) + G^* = v + \sum_{n=1}^{N} H_n \Delta S_n^*,$$

  we can transform the previous optimization problem with contraints to an unconstrained one.

- That is,

Unconstrained Optimal Portfolio Problem $(\mathbf{UOPP}(v, U))$

$$\max_{(H_1,\ldots,H_N)^T \in \mathbb{R}^N} \mathbb{E}\left[U\left(B(1) \left\{ v + \sum_{n=1}^{N} H_n \Delta S_n^* \right\} \right) \right] \qquad (2)$$

- Note that we just have moved the inital wealth v from the constrain to the functional to optimize, eliminating the constraint and reducing the arguments of the functional by one.

Optimal portfolio problem and arbitrage opportunities

- Given a solution to **UOPP**(v, U) we get a solution to **OPP**(v, U) using $v = H_0 B(0) + \sum_{n=1}^{N} H_n S_n(0)$, and viceversa.

$\exists$ solution to the **OPP**$(v, U) \Longrightarrow \nexists$ **AO**.

Optimal portfolio problem and RNPM

Suppose H is a solution to the **OPP**(v, U) and $V(1)$ is its final value. Then,

$$Q(\omega) = \frac{B(1,\omega)\,U'(V(1,\omega),\omega)}{\mathbb{E}\left[B(1)\,U'(V(1))\right]} P(\omega), \quad \omega \in \Omega,$$

is a **RNPM**.

State price density

Let $Q \in \mathbb{M}$, then $L = Q/P$ is called the **state price density/ vector** (associated to Q). Suppose $B(1) = B(0)(1+r)$ is constant, H is a solution to the **OPP**(v, U) and $V(1)$ is its final value. Then,

$$L(\omega) = \frac{Q(\omega)}{P(\omega)} = \frac{U'(V(1,\omega),\omega)}{\mathbb{E}[U'(V(1))]}, \quad \omega \in \Omega,$$

that is, the state price density is proportional to the marginal utility of the terminal wealth ($U'(V(1))$).

Viability of a market

- If there exists a **RNPM** Q, then does the **OPP**(v, U) have a solution?
- Not necessarily, for some v and U it may happen that **OPP**(v, U) does not have a solution.
- However, one can always find a pair (v, U) such that **OPP**(v, U) has a solution.

Viability of a market

A market model is **viable** if there exists a function $U : \mathbb{R} \times \Omega \to \mathbb{R}$ and an initial wealth v such that $u \mapsto U(u, \omega)$ is concave, strictly increasing and differentiable for each $\omega \in \Omega$ and such that the corresponding **OPP**(v, U) has a solution.

Example of OPP

Example

- Take a generic market model with $N = 2$ and $K = 3$.
- Consider the utility function $U(u) = -e^{-u}$, with derivative $U'(u) = e^{-u}$.
- Then, at a maximum the following equation must hold

$$\begin{aligned} 0 &= \frac{\partial}{\partial H_1}\mathbb{E}\left[U\left(B(1)\left\{v + H_1\Delta S_1^* + H_2\Delta S_2^*\right\}\right)\right] \\ &= \mathbb{E}\left[\Delta S_1^* \exp\left(-B(1)\left\{v + H_1\Delta S_1^* + H_2\Delta S_2^*\right\}\right)\right], \\ 0 &= \frac{\partial}{\partial H_2}\mathbb{E}\left[U\left(B(1)\left\{v + H_1\Delta S_1^* + H_2\Delta S_2^*\right\}\right)\right] \\ &= \mathbb{E}\left[\Delta S_2^* \exp\left(-B(1)\left\{v + H_1\Delta S_1^* + H_2\Delta S_2^*\right\}\right)\right]. \end{aligned}$$

- One has to solve a system of nonlinear equations for H_1 and H_2 (numerical methods).

Risk Neutral Computational Approach to the OPP

Risk neutral computational approach

- The previous example shows that the direct approach to solve the **OPP** easily leads to computational difficultites (system of nonlinear equations)
- There is a more efficient approach based on **RNPM**.
- Recall that we want to solve

Optimal Portfolio Problem $(\mathrm{OPP}(v, U))$

$$\left.\begin{array}{cc} \max & \mathbb{E}\left[U\left(V\left(1\right)\right)\right] \\ \text{subject to} & V\left(0\right) = v \in \mathbb{R}, \\ & H \in \mathbb{R}^{N+1}, \end{array}\right\}$$

Risk neutral computational approach

- The risk neutral computational approach consists in two steps:

Step 1 Maximize $\mathbb{E}\left[U\left(V\left(1\right)\right)\right]$ over the subset of feasible random variables $V\left(1\right)$. That is , determine the optimal terminal wealth $V\left(1\right)$ such that $V\left(0\right)=v$.

Step 2 Given the optimal terminal wealth $V\left(1\right)$, determine a trading strategy H that generates it.

Risk neutral computational approach

- **Step 2** is easy. It boils down to solve a system of linear equations. That is, given $V(1) \in \mathbb{R}^K$, find $H \in \mathbb{R}^{N+1}$ such that

$$\left.\begin{array}{rcl} H_0 B(1) + \sum_{n=1}^{N} H_n S_n(1, \omega_1) & = & V(1, \omega_1) \\ & \vdots & \\ H_0 B(1) + \sum_{n=1}^{N} H_n S_n(1, \omega_1) & = & V(1, \omega_K), \end{array}\right\}$$

- **Step 1** is more challenging and relies on finding a "convenient" feasible region, which we will denote by $\mathbb{W}_v$. Besides this, it is a straightforward optimization problem.

Risk neutral computational approach

- From now on we assume that the model arbitrage free and complete, i.e., $\mathbb{M} = \{Q\}$.
- In this case the set of feasible/attainable wealths is given by

$$\mathbb{W}_v = \left\{ W \in \mathbb{R}^K : \mathbb{E}_Q \left[\frac{W}{B(1)} \right] = v \right\}$$

- Note that, for any trading strategy H with $V(0) = v$ we have, by the risk neutral pricing principle, that

$$\mathbb{E}_Q [V(1)/B(1)] = V(0) = v.$$

- Conversely, for any $W \in \mathbb{W}_v$ there exists, by the completeness and the risk neutral pricing principle, an H such that $V(0) = v$ and $V(1) = W$.
- The subproblem to solve in **Step 1** is

$$\max_{W \in \mathbb{W}_v} \mathbb{E}[U(W)].$$

Risk neutral computational approach

- The previous subproblem is a contrained optimization problem, with equality constraints.
- To solve it, we will use the Lagrange multiplier method
- Consider the Lagrangian function

$$\mathcal{L}(W;\lambda) = \mathbb{E}[U(W)] - \lambda\left(\mathbb{E}_Q\left[\frac{W}{B(1)}\right] - v\right).$$

- Using the state price density $L = Q/P$ we get

$$\begin{aligned}\mathcal{L}(W;\lambda) &= \mathbb{E}[U(W)] - \lambda\left(\mathbb{E}\left[L\frac{W}{B(1)}\right] - v\right)\\ &= \mathbb{E}\left[U(W) - \lambda\left(L\frac{W}{B(1)} - v\right)\right]\\ &= \sum_{k=1}^{K}\left\{U(W_k,\omega_k) - \lambda L(\omega_k)\frac{W_k}{B(1,\omega_k)} + \lambda v\right\}P(\omega_k),\end{aligned}$$

where $W_k := W(\omega_k)$.

Risk neutral computational approach

- The first order optimality conditions gives

$$0 = \frac{\partial}{\partial \lambda} L(W;\lambda) = -\left(\mathbb{E}_Q\left[\frac{W}{B(1)}\right] - v\right) \Longleftrightarrow \mathbb{E}_Q\left[\frac{W}{B(1)}\right] = v,$$

$$0 = \frac{\partial}{\partial W_k} L(W;\lambda) = \left\{U'(W_k,\omega_k) - \lambda \frac{L(\omega_k)}{B(1,\omega_k)}\right\} P(\omega_k),$$

  where $k = 1, ..., K$.
- Since $U(\cdot,\omega)$ is concave, $U'(\cdot,\omega)$ is decreasing and the inverse of $U'(\cdot,\omega)$ exists, for each $\omega \in \Omega$ fixed.
- Let $I(\cdot,\omega)$ denote the inverse of $U'(\cdot,\omega)$.

Risk neutral computational approach

- A solution $\left(\widehat{W}, \widehat{\lambda}\right)$ of the previous equations is given by $\widehat{W} = I\left(\widehat{\lambda}L/B(1)\right)$, that is,

$$\widehat{W}_k = I\left(\frac{\widehat{\lambda}L(\omega_k)}{B(1, \omega_k)}\right), \qquad k = 1, ..., K,$$

and $\widehat{\lambda}$ is chosen such that

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B(1)}\right] = \mathbb{E}_Q\left[\frac{I\left(\widehat{\lambda}L/B(1)\right)}{B(1)}\right]$$
$$= \sum_{k=1}^{K} \frac{I\left(\widehat{\lambda}L(\omega_k)/B(1, \omega_k)\right)}{B(1, \omega_k)} Q(w_k).$$

- The function I is decreasing and its range will normally include $(0, +\infty)$, so $\widehat{\lambda}$ satisfying the previous equation will exist for $v > 0$.

Risk neutral computational approach

Example

- Consider a market with $N = 2, K = 3, B(0) = 1, B(1) = \frac{10}{9}, S_1^*(0) = 6, S_2^*(0) = 10$, and with payoff matrix

$$S^*(1,\Omega) = \begin{pmatrix} 1 & 6 & 13 \\ 1 & 8 & 9 \\ 1 & 4 & 8 \end{pmatrix}.$$

- We will solve the **OPP** with utility function $U(u) = -e^{-u}$.
- This example is discussed in the smartboard.

Thank you!