Review of Probability

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Information and Measurability

Information and measurability

- Our standing assumption is that $\#\Omega = K < \infty$.

Outcomes of an experiment $\omega_1, ..., \omega_K$ are called **elementary events** or **sample points** and the finite set $\Omega = \{\omega_1, ..., \omega_K\}$ is called the **space of of elementary events** or the **sample space**. **Events** are all subsets $A \subseteq \Omega$ for which, under the conditions of the experiment, one can conclude that either “the outcome $\omega \in A$” or “the outcome $\omega \notin A$”.

Information and measurability

Example

- The random experiment consists in tossing a coin three times.
- Then, $\#\Omega = 8$ and

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

- Event = " 2 heads in all " $= \{HHT, HTH, THH\} \subset \Omega$.

A collection $\mathcal{F}$ of subsets of Ω is called an **algebra** on Ω if

1. $\Omega \in \mathcal{F}$.
2. $A \in \mathcal{F} \Rightarrow A^c := \Omega \setminus A \in \mathcal{F}$.
3. $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

Information and measurability

- Note that $\emptyset = \Omega^c \in \mathcal{F}$ and

$$A, B \in \mathcal{F} \Rightarrow A \cap B = (A^c \cup B^c)^c \in \mathcal{F}.$$

  Hence, an algebra $\mathcal{F}$ is a family of subsets of Ω which is closed under complementation and finitely many set operations (intersection and union).
- If $\#\Omega = \infty$, we need the closedness property to hold for infinitely many set operations.
- In this case, we say that a collection $\mathcal{F}$ of subsets of Ω is a σ**-algebra** on Ω if 1., 2. and

  $3'$. $\{A_n\}_{n\geq 1} \subseteq \mathcal{F} \Rightarrow \bigcup_{n\geq 1} A_n \in \mathcal{F}$.
- For Ω with $\#\Omega < \infty$ both concepts coincide.

Information and measurability

Example

Consider the following examples

1. $\mathcal{F}_1 = \{\emptyset, \Omega\}$ trivial algebra. (contains no information)
2. $\mathcal{F}_2 = \mathcal{P}(\Omega)$ collection of all subsets of Ω. (contains all the information)
3. $\mathcal{F}_3 = \{\emptyset, \Omega, A, A^c\}$ algebra generated by the event A. (contains the minimal information needed to decide if A has occurred or not)

Information and measurability

Let S be a class of subsets of Ω. Then $\mathfrak{a}(S)$, the **algebra generated by** S, is the smallest algebra on Ω containing S. That is,

1. $S \subseteq \mathfrak{a}(S)$,
2. If $S \subseteq \mathcal{F}$, where $\mathcal{F}$ is an algebra, then $S \subseteq \mathfrak{a}(S) \subseteq \mathcal{F}$.

Note that

- If $S_1 \subseteq S_2$ then $\mathfrak{a}(S_1) \subseteq \mathfrak{a}(S_2)$.
- The intersection of an arbitrary number of algebras is an algebra.
- $\mathfrak{a}(S)$ is the intersection of all the algebras on Ω containing S.

Information and measurability

Example

Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

1. $S_1 = \{\{\omega_1\}\}$, then
$$\mathfrak{a}(S_1) = \{\Omega, \emptyset, \{\omega_1\}, \{\omega_2, \omega_3, \omega_4\}\}.$$
2. $S_2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}$, then
$$\mathfrak{a}(S_2) = \{\Omega, \emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_2, \omega_3, \omega_4\}, \{\omega_1, \omega_4\}, \{\omega_1, \omega_2, \omega_3\}\}.$$
3. $S_3 = \{\{\omega_1\}, \{\omega_1, \omega_4\}\}$, then
$$\mathfrak{a}(S_3) = \{\Omega, \emptyset, \{\omega_1\}, \{\omega_1, \omega_4\}, \{\omega_2, \omega_3, \omega_4\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}.$$

Information and measurability

Example 3

- Since $S_1 \subseteq S_2$, we have that $\mathfrak{a}(S_1) \subseteq \mathfrak{a}(S_2)$.
- The algebra $\mathfrak{a}(S_2)$ contains the events in $\mathfrak{a}(S_1)$ and more.
- Hence, $\mathfrak{a}(S_2)$ is more informative than $\mathfrak{a}(S_1)$.
- Note that, $S_2 \nsubseteq S_3$ and $S_3 \nsubseteq S_2$, but $\mathfrak{a}(S_2) = \mathfrak{a}(S_3)$ and, therefore, $\mathfrak{a}(S_2)$ and $\mathfrak{a}(S_3)$ contain the same information.

Information and measurability

An interesting class of subsets of Ω are those which form a partition of Ω. A class of subsets $\pi = \{A_1, \ldots, A_m\}$ of Ω is a **partition** of Ω if

1. $A_i \cap A_j = \emptyset, \qquad i \neq j,$
2. $\cup_{i=1}^{m} A_i = \Omega.$

Given two partitions π_1, π_2 of Ω, we say that π_2 **is finer than (or refines)** π_1, if for any $A \in \pi_2$ there exists $B \in \pi_1$ such that $A \subseteq B$ and we will denote it by $\pi_1 \subseteq \pi_2$.

Information and measurability

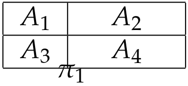
Given two partitions π_1, π_2 of Ω, we may define its **intersection** $\pi_1 \cap \pi_2$ to be the following partition

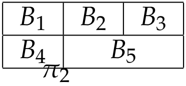
$$\pi_1 \cap \pi_2 = \{A \cap B : A \in \pi_1 \text{ and } B \in \pi_2\}.$$

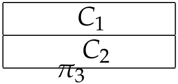
Note that, in general, neither $\pi_1 \subseteq \pi_2$ nor $\pi_2 \subseteq \pi_1$, but $\pi_1 \subseteq \pi_1 \cap \pi_2$ and $\pi_2 \subseteq \pi_1 \cap \pi_2$.

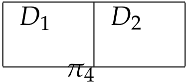
Information and measurability

Example

- 

$\subseteq$


- 

neither $\subseteq$ nor $\supseteq$


But $\pi_3 \cap \pi_4 = \pi_1$ and $\pi_3 \subseteq \pi_1$, $\pi_4 \subseteq \pi_1$.

Information and measurability

Why are partitions interesting?

- For any algebra $\mathcal{F}$ on Ω, there exists a partition π such that $\mathcal{F} = \mathfrak{a}(\pi)$ (**bijection**).
- The elements of $\mathfrak{a}(\pi)$ are all possible unions of the elements in π. (**easy structure**)
- Let $X : \Omega \to \{x_1, \ldots, x_M\}$, where $M \leq K = \#\Omega$, represent a measurement in a random experiment. Then, the following class of subsets of Ω is a partition

$$\pi_X = \left\{ X^{-1}(x_i) = \{\omega \in \Omega : X(\omega) = x_i\}, i = 1, ..., M \right\}.$$

(**easy to interpret**)

Information and measurability

Let $\mathcal{F}$ be an algebra on Ω. We say that function $X : \Omega \to \{x_1, \ldots, x_M\}$ is $\mathcal{F}$**-measurable** (measurable with respect to $\mathcal{F}$) if

$$X^{-1}(x_i) = \{\omega \in \Omega : X(\omega) = x_i\} \in \mathcal{F}, \quad i = 1, ..., M.$$

X is a random variables if and only if X is $\mathcal{P}(\Omega)$-measurable. The **algebra generated by a finite number of r.v.** $X_1, X_2, \ldots, X_n$, denoted by $\mathfrak{a}(X_1, X_2, \ldots, X_n)$, is defined as $\mathfrak{a}\left(\bigcap_{i=1}^n \pi_{X_i}\right)$.

Information and measurability

- $\mathfrak{a}(X) = \mathfrak{a}(\pi_X)$ is the smallest algebra $\mathcal{F}$ such that X is $\mathcal{F}$-measurable.
- Let $\mathcal{F} = \mathfrak{a}(\pi)$ where π is a partition of Ω. Then, X is $\mathcal{F}$-measurable if and only if X is constant on each element of the partition π.
- Usually, $\mathcal{P}(\Omega)$ is strictly finer than $\mathfrak{a}(X)$, that is, by observing X we cannot get all the information available in the sample space Ω.
- $\mathfrak{a}(X) = \mathcal{P}(\Omega)$ if and only if X takes $K = \#\Omega$ different values.

Information and measurability

Example

- Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- Consider the random variables

$$X(\omega) = \begin{cases} 2 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & \text{if} \quad \omega = \omega_1 \\ 2 & \text{if} \quad \omega = \omega_2 \\ 3 & \text{if} \quad \omega = \omega_3 \\ 4 & \text{if} \quad \omega = \omega_4 \end{cases}.$$

- Then,

$$\pi_X = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\},$$
$$\mathfrak{a}(X) = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\},$$

Information and measurability

Example 5

$$\pi_Y = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\},$$
$$\mathfrak{a}(Y) = \mathfrak{a}(\pi_Y) = \mathcal{P}(\Omega).$$

- Let Z be the “random variable” $Z \equiv 1$.
- Then, $\pi_Z = \{\Omega\}$ and $\mathfrak{a}(Z) = \mathfrak{a}(\pi_Z) = \{\emptyset, \Omega\}$.
- Note that Z (in fact any constant random variable) is measurable with respect to any algebra on Ω.

Information and measurability

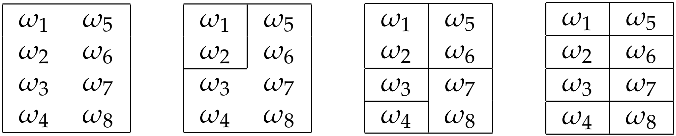
A **filtration** $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\ldots,T}$ on Ω is a sequence of algebras on Ω such that $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, $t = 0, \ldots, T$.

- We will always assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and usually $\mathcal{F}_T = \mathcal{P}(\Omega)$.
- A filtration models the evolution of the information at our disposal through time.
- At time $t = 0$ we have no information and at time T, if $\mathcal{F}_T = \mathcal{P}(\Omega)$, we have full information.

Information and measurability

Two graphical ways to represent the flow of information:

- Partitions



- Trees

Information and measurability

A **stochastic process** $X = \{X(t)\}_{t=0,\dots,T}$ is a collection of random variables indexed by $t = 0, \dots, T$. You can see it as a function $X : \Omega \times \{0, \dots, T\} \to \mathbb{R}$ or as random variable $X : \Omega \to \mathbb{R}^{\{0,\dots,T\}}$, where $\mathbb{R}^{\{0,\dots,T\}}$ denotes the set of all real-valued functions with domain of definition $\{0, \dots, T\}$. We say that a stochastic process X is **adapted to the filtration** $\mathbb{F}$ or $\mathbb{F}$**-adapted** if X_t is $\mathcal{F}_t$-measurable, $t = 0, \dots, T$.

Information and measurability

The **natural filtration generated by a stochastic process** X, denoted by $\mathbb{F}^X$, is defined by

$$\mathbb{F}^X = \left\{ \mathcal{F}_t^X = \mathfrak{a}\left(X\left(0\right), X\left(1\right), \ldots, X\left(t\right)\right) \right\}_{t=0,\ldots,T}.$$

- $\mathbb{F}^X$ is the minimal filtration to which X is adapted to. It contains the information that you can get by observing the process X.

We say that a process $X = \{X\left(t\right)\}_{t=1,\ldots,T}$ is **predictable with respect to a filtration** $\mathbb{F}$ or $\mathbb{F}$**-predictable** if X_t is $\mathcal{F}_{t-1}$-measurable, $t = 1, \ldots, T$.

Information and measurability

Example

- Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $X = \{X(t)\}_{t=0,1,2}$ with $X(0) = 3$,

$$X(1,\omega) = \begin{cases} 5 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 2 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases},$$

$$X(2,\omega) = \begin{cases} 6 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 3 & \text{if} \quad \omega = \omega_3 \\ 2 & \text{if} \quad \omega = \omega_4 \end{cases}.$$

- Then,

$$\mathcal{F}_0^X = \mathfrak{a}(X(0)) = \mathfrak{a}\left(\pi_{X(0)}\right) = \{\emptyset, \Omega\},$$

Information and measurability

Example 6

$$\mathcal{F}_1^X = \mathfrak{a}\left(X\left(0\right), X\left(1\right)\right) = \mathfrak{a}\left(\pi_{X(0)} \cap \pi_{X(1)}\right) = \mathfrak{a}\left(\pi_{X(1)}\right)$$
$$= \mathfrak{a}\left(\left\{\left\{\omega_1, \omega_2\right\}, \left\{\omega_3, \omega_4\right\}\right\}\right) = \left\{\emptyset, \Omega, \left\{\omega_1, \omega_2\right\}, \left\{\omega_3, \omega_4\right\}\right\},$$
$$\mathcal{F}_2^X = \mathfrak{a}\left(X\left(0\right), X\left(1\right), X\left(2\right)\right) = \mathfrak{a}\left(\pi_{X(0)} \cap \pi_{X(1)} \cap \pi_{X(2)}\right)$$
$$= \mathfrak{a}\left(\pi_{X(2)}\right) = \mathfrak{a}\left(\left\{\left\{\omega_1, \omega_2\right\}, \left\{\omega_3\right\}, \left\{\omega_4\right\}\right\}\right)$$
$$= \left\{\emptyset, \Omega, \left\{\omega_1, \omega_2\right\}, \left\{\omega_3\right\}, \left\{\omega_4\right\}, \left\{\omega_1, \omega_2, \omega_3\right\}, \left\{\omega_1, \omega_2, \omega_4\right\}, \left\{\omega_3, \omega_4\right\}\right\}.$$

- In this case $\mathcal{F}_2^X \neq \mathcal{P}\left(\Omega\right)$.
- Check what happens if $X\left(2, \omega_2\right) = 3$.

Information and measurability

- The systematic way to compute $\mathfrak{a}(S)$, where $S \subseteq \mathcal{P}(\Omega)$, is to identify the finest partition of Ω that you can obtain by basic set operations on all elements of S, denoted by π_S.
- Then, the elements of $\mathfrak{a}(S)$ will be all possible unions of elements in π_S.

Conditional Expectation

Conditional expectation

- Recall that a **probability measure** P on a finite sample space $\Omega = \{\omega_1, \ldots, \omega_K\}$ is a function $P : \Omega \to [0, 1]$ such that $\sum_{i=1}^{K} P(\omega_i) = 1$.
- The triple $(\Omega, \mathcal{P}(\Omega), P)$ is a **probability space**.
- In addition, we will assume that $P(\omega_i) > 0$, $i = 1, \ldots, K$. This assumption is not essential but implies that all sets in $\mathcal{P}(\Omega)$ have strictly positive probability, which simplifies the statements about conditional probabilities and conditional expectations.
- Given an event $A \in \mathcal{P}(\Omega)$ the **probability of** A happening is given by

$$P(A) = \sum_{\omega \in A} P(\omega).$$

Condtional expectation

- We say that **two events** $A, B \in \mathcal{P}(\Omega)$ **are independent** if

$$P(A \cap B) = P(A) P(B) .$$

- Given two events $A, B \in \mathcal{P}(\Omega)$, the **probability of** A **given** B, denoted by

$$P(A|B) = P(A \cap B) / P(B) .$$

In general, we would need to assume that $P(B) > 0$ for this probability to be well defined. However, thanks to the assumption on the strict positivity of P, this probability is always well defined in our setup.

Conditional expectation

Given **two algebras** $\mathcal{F}_1, \mathcal{F}_2$ on Ω we say that they **are independent** if for all $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ we have that A and B are independent. Given a random variable X we define its **expectation** by

$$\mathbb{E}\left[X\right] = \sum_{\omega \in \Omega} X\left(\omega\right) P\left(\omega\right).$$

Conditional expectation. Definition.

Given an algebra $\mathcal{F}$ and a random variable X we define the **conditional expectation of X given $\mathcal{F}$** as the unique random variable Z, denoted by $\mathbb{E}\left[X|\mathcal{F}\right]$, satisfying

1. Z is $\mathcal{F}$-measurable.
2. $\mathbb{E}\left[\mathbf{1}_A X\right] = \mathbb{E}\left[\mathbf{1}_A Z\right]$, $A \in \mathcal{F}$.

- Note that since $\mathbb{E}\left[X|\mathcal{F}\right]$ is $\mathcal{F}$-measurable, it is constant on the partition that generates $\mathcal{F}$.
- How we compute $\mathbb{E}\left[X|\mathcal{F}\right]$?

Conditional expectation

Let $A \in \mathcal{P}(\Omega)$ and X be a random variable. Then, the **conditional expectation of X given A** is the quantity

$$\mathbb{E}[X|A] = \sum_x x P(X = x|A),$$

where x are the values taken by X and

$$P(X = x|A) = \frac{P(\{\omega : X(\omega) = x\} \cap A)}{P(A)}.$$

- A remark analogous to Remark 28 applies to the previous definition.

Conditional expectation

Let $\mathcal{F}$ be an algebra on Ω, X be a random variable and let $\pi = \{A_1, \ldots, A_m\}$ be the partition of Ω such that $\mathcal{F} = \mathfrak{a}(\pi)$. Then,

$$\mathbb{E}\left[X \middle| \mathcal{F}\right](\omega) = \sum_{i=1}^{m} \mathbb{E}\left[X \middle| A_i\right] \mathbf{1}_{A_i}(\omega).$$

Conditional expectation

- Usually we are given (or we guess) a candidate Z to be $\mathbb{E}\left[X|\,\mathcal{F}\right]$, then we need to check conditions $1)$ and $2)$ in Definition.
- When $\mathcal{F} = \sigma\left(\pi\right)$, π a partition it suffices to check that the candidate Z is constant over the elements of π ($\mathcal{F}$-measurable) and check condition $2)$ in Definition only for $A_i \in \pi$.

Conditional expectation

Example

- Let $\Omega = \{\omega_1, \ldots, \omega_4\}$ and $P(\omega_i) = 1/4, i = 1, ..., 4.$
- Consider the algebra $\mathcal{F} = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and the random variable X given by

$$X(\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2, \omega_3 \\ 3 & \text{if} \quad \omega = \omega_4 \end{cases}$$
$$= 9\mathbf{1}_{\{\omega_1\}}(\omega) + 6\mathbf{1}_{\{\omega_2,\omega_3\}}(\omega) + 3\mathbf{1}_{\{\omega_4\}}(\omega).$$

- Compute $\mathbb{E}[X|\mathcal{F}]$.

Conditional expectation

Suppose X and Y are random variables on $(\Omega, \mathcal{P}(\Omega), P)$, $\mathcal{G}$ is an algebra on Ω, $a, b \in \mathbb{R}$. Then,

1. ***Linearity***: $\mathbb{E}[aX + bY | \mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$.
2. ***Law of total expectation***: $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$.
3. ***Independence***: If X is independent of $\mathcal{G}$ then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.
4. ***Measurability***: If Y is $\mathcal{G}$-measurable then $\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]$.
5. ***Tower property***: If $\mathcal{H}$ is an algebra on Ω such that $\mathcal{H} \subseteq \mathcal{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$.

Conditional expectation. Theorem

Let X be a random variable on $(\Omega, \mathcal{P}(\Omega), P)$ and $\mathcal{G}$ an algebra on Ω. Then,

$$\mathbb{E}[X|\mathcal{G}] = \arg\min \left\{ \mathbb{E}\left[(X - Y)^2\right] : Y \text{ being } \mathcal{G}\text{-measurable} \right\}.$$

The conditional expectation is the best prediction of X based on the information contained in $\mathcal{G}$, in the sense of minimizing the L^2 error (variance).

Conditional expectation. Definition

Let $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\dots,T}$ be a filtration on $(\Omega, \mathcal{P}(\Omega), P)$. A stochastic process $X = \{X(t)\}_{t=0,\dots,T}$ is a **($\mathbb{F}$-) martingale** if

1. X is $\mathbb{F}$-adapted.
2. For $t \in \{0, \dots, T\}$,$s \geq 0$,$t + s \in \{0, \dots, T\}$ we have

$$\mathbb{E}\left[X(t+s) \middle| \mathcal{F}_t \right] = X(t).$$

- Intuitively, the best forecast of the process at some future time $t + s$ given today's information $\mathcal{F}_t$ is the value of the process today.

Conditional expectation

- An $\mathbb{F}$-adapted process X is called a **(sub) supermartingale** if

$$\mathbb{E}\left[X(t+s)|\mathcal{F}_t\right](\geq) \leq X(t).$$

- If $\#\Omega = +\infty$ then we need to impose that $\mathbb{E}\left[|X(t)|\right] < \infty$ for all $t = 0, \ldots, T$.
- In the previous definitions we can change $X(t+s)$ by $X(t+1)$.

Conditional expectation. Proposition

[Martingale transform or stochastic integral] Let $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\ldots,T}$ be a filtration on $(\Omega, \mathcal{P}(\Omega), P)$. Let H be an $\mathbb{F}$-predictable process and M an $\mathbb{F}$-martingale. Then, the process Y defined by $Y_0 = c$ (a constant) and

$$\begin{aligned} Y(t) &= \sum_{s=1}^{t} H(s)\left(M(s) - M(s-1)\right) \\ &= \sum_{s=1}^{t} H(s)\,\Delta M(s), \qquad t = 1, \ldots, T, \end{aligned}$$

is an $\mathbb{F}$-martingale with $\mathbb{E}\left[Y(t)\right] = c$.

Thank you!