Multiperiod Financial Markets

STK-MAT 3700/4700 An Introduction to Mathematical Finance

O. Tymoshenko

University of Oslo
Department of Mathematics

Oslo 2022.10.31

Contents

Model Specifications

Model specifications

A **multiperiod model of financial markets** *is specified by the following ingredients:*

1. $T+1$ trading dates: $t=0,\dots,T$.
2. A finite probability space $(\Omega,\mathcal{P}(\Omega),P)$ with $\#\Omega=K$ and $P(\omega)>0, \omega\in\Omega$.
3. A filtration $\mathbb{F}=\{\mathcal{F}_t\}_{t=0,\dots,T}$.
4. A bank account process $B=\{B(t)\}_{t=0,\dots,T}$ with $B(0)=1$ and $B(t,\omega)>0$, $t\in\{0,\dots,T\}$ and $\omega\in\Omega$. B is assumed to be an $\mathbb{F}$-adapted process.
5. N risky asset processes $S_n=\{S_n(t)\}_{t=0,\dots,T}$, where S_n is a nonnegative $\mathbb{F}$-adapted stochastic process for each $n=1,\dots,N$.

Model specifications

- The filtration $\mathbb{F}$ represents the information available to the traders.
- In this course we will take $\mathbb{F}$ to be equal to $\mathbb{F}^{B,S}$, that is, the filtration generated by the bank account process and the N risky asset processes:

$$\mathcal{F}_t = \mathfrak{a}\left(\{B(u), S_1(u), ..., S_N(u)\}_{u \le t}\right), \qquad t = 0, ..., T.$$

- The bank account process B is nondecreasing, which implies

$$r(t) = (B(t) - B(t-1))/B(t-1) \ge 0, \qquad t = 1, ..., T$$

- When $r(t) = r, t = 1, ..., T$, then $B(t) = (1+r)^t$, $t = 1, ..., T$ and

$$\mathcal{F}_t = \mathfrak{a}\left(\{S_1(u), ..., S_N(u)\}_{u \le t}\right), \qquad t = 0, ..., T$$

Model specifications

A **trading strategy** $H = (H_0, H_1, ..., H_N)^T$ is a vector of stochastic processes $H_n = \{H_n(t)\}_{t=1,...,T}$, which are predictable with respect to $\mathbb{F}$. That is,

$$H_n(t) \text{ are } \mathcal{F}_{t-1}\text{-measurable}, \qquad n = 0, ..., N, \quad t = 1, ..., T.$$

Model specifications

- Note that $H_n, n = 0, ..., N$, being $\mathbb{F}$-predictable processes, they are also $\mathbb{F}$-adapted processes.
- $H_n(0), n = 0, ..., N$ is not specified because:
  - $H_n(t), n \geq 1$ is the number of shares of the nth risky asset that the investor own from time $t-1$ to time t.
  - $H_0(t) B(t-1)$ is the amount of money that the trader invest/borrow in the money market (bank account) from time $t-1$ to time t.
- The trading position $H_n(t)$ is decided by the trader at time $t-1$ and then he/she only has the information associated to $\mathcal{F}_{t-1} \Rightarrow H_n(t)$ are $\mathbb{F}$-predictable.

Model specifications

The **value process** $V = \{V(t)\}_{t=0,\dots,T}$ is the stochastic process defined by

$$V(t) = \begin{cases} H_0(1)\,B(0) + \sum_{n=1}^{N} H_n(1)\,S_n(0) & \text{if} \quad t = 0, \\ H_0(t)\,B(t) + \sum_{n=1}^{N} H_n(t)\,S_n(t) & \text{if} \quad t \geq 1. \end{cases} \tag{1}$$

The **gains process** $G = \{G(t)\}_{t=1,\dots,T}$ is the stochastic process defined by

$$G(t) = \sum_{u=1}^{t} H_0(u)\,\Delta B(u) + \sum_{n=1}^{N} \sum_{u=1}^{t} H_n(u)\,\Delta S_n(u), \qquad t \geq 1, \tag{2}$$

where $\Delta B(u) = B(u) - B(u-1)$ and $\Delta S_n(u) = S_n(u) - S_n(u-1)$.

Model specifications

- Both V and G are $\mathbb{F}$-adapted processes.
- $H_n(t)\,\Delta S_n(t)$ represents the one-period gain or loss due to owning $H_n(t)$ shares of the security n between times $t-1$ and t.
- $G(t)$ represents the cumulative gain or loss up to time t of the portfolio.
- $V(t)$ represents the time-t value of the portfolio `before` any transactions (changes in H) are made at time t.

Model specifications

Remark 9

- The time-t value of the portfolio just after any time-t transactions are made is

$$H_0(t+1)\,B(t) + \sum_{n=1}^{N} H_n(t+1)\,S_n(t), \qquad t \geq 1. \tag{3}$$

- In general these two portfolio values can be different, which means that we add or withdraw some money from the portfolio.
- If we do not allow this possibility we have a self-financing portfolio.

Model specifications

Definition

A trading strategy H is **self-financing** if

$$V(t) = H_0(t+1)\,B(t) + \sum_{n=1}^{N} H_n(t+1)\,S_n(t), \qquad t = 1, ..., T-1. \tag{4}$$

- It is easy to check that H is self-financing if and only if

$$V(t) = V(0) + G(t), \qquad t = 1, ..., T. \tag{5}$$

- If no money is added or withdrawn from the portolio between time $t = 0$ and $t = T$, then any change in the portfolio's value is due to gain or loss in the investments (changes in the prices of the assets).

Model specifications

Definition

- The **discounted price process** $S_n^* = \{S_n^*(t)\}_{t=0,...,T}$ is defined by

$$S_n^*(t) = \frac{S_n(t)}{B(t)}, \qquad t = 0, ..., T, \quad n = 1, ..., N. \tag{6}$$

- The **discounted value process** $V^* = \{V^*(t)\}_{t=0,...,T}$ is defined by

$$V^*(t) = \begin{cases} H_0(1) + \sum_{n=1}^{N} H_n(1) S_n^*(0) & \text{if} \quad t = 0, \\ H_0(t) + \sum_{n=1}^{N} H_n(t) S_n^*(t) & \text{if} \quad t \geq 1. \end{cases} \tag{7}$$

Model specifications

Definition ??

- The **discounted gains process** $G^* = \{G^*(t)\}_{t=1,\dots,T}$ is defined by

$$G^*(t) = \sum_{n=1}^{N} \sum_{u=1}^{t} H_n(u)\, \Delta S_n^*(u), \quad t = 1, \dots, T, \tag{8}$$

  where $\Delta S_n^*(u) = S_n^*(u) - S_n^*(u-1)$.
- It is easy to check that a trading strategy H is self-financing if and only if

$$V^*(t) = V^*(0) + G^*(t), \qquad t = 0, \dots, T \tag{9}$$

Model specifications

Example

- Consider a market with $N=1$,$K=4$, $B(t)=(1+r)^t$, $r\geq 0$, $S(0)=5$,

$$S(1,\omega)=\begin{cases} 8 & \text{if} \quad \omega=\omega_1,\omega_2 \\ 4 & \text{if} \quad \omega=\omega_3,\omega_4 \end{cases}$$
$$=8\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega)+4\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega),$$
$$S(2,\omega)=\begin{cases} 9 & \text{if} \quad \omega=\omega_1 \\ 6 & \text{if} \quad \omega=\omega_2,\omega_3 \\ 3 & \text{if} \quad \omega=\omega_4 \end{cases}$$
$$=9\mathbf{1}_{\{\omega_1\}}(\omega)+6\mathbf{1}_{\{\omega_2,\omega_3\}}(\omega)+3\mathbf{1}_{\{\omega_4\}}(\omega).$$

- Compute the filtration generated by S.
- Write down a generic H,V and G.

Economic Considerations

Economic considerations

An **arbitrage opportunity** is a trading strategy H such that

1. H is self-financing.
2. $V(0) = 0$.
3. $V(T) \geq 0$.
4. $\mathbb{E}[V(T)] > 0$.

Alternative equivalent formulations:

Alternative 1

H is an arbitrage opportunity if

1. H is self-financing.

b) $V^*(0) = 0$.

c) $V^*(T) \geq 0$.

d) $\mathbb{E}[V^*(T)] > 0$.

Alternative 2

H is an arbitrage opportunity if

1. H is self-financing.

b) $V^*(0) = 0$.

c') $G^*(T) \geq 0$.

d') $\mathbb{E}[G^*(T)] > 0$.

Economic considerations

A **risk neutral probability measure (martingale measure)** is a probability measure Q such that

1. $Q(\omega) > 0, \omega \in \Omega$.
2. S_n^*, $n = 1, ..., N$ are martingales under Q, that is,

$$\mathbb{E}_Q\left[S_n^*(t+s) \middle| \mathcal{F}_t \right] = S_n^*(t), \qquad t, s \geq 0, n = 1, ..., N. \tag{10}$$

- It suffices to check (10) for $s = 1$ and $t = 0, ..., T-1$, that is,

$$\mathbb{E}_Q\left[S_n^*(t+1) \middle| \mathcal{F}_t \right] = S_n^*(t).$$

- If $B(t) = (1+r)^t$, then (10) is equivalent to

$$\mathbb{E}_Q\left[S_n(t+1) \middle| \mathcal{F}_t \right] = (1+r) S_n(t). \tag{11}$$

Economic considerations

Example (Continuation of Example 1)

- Consider a market with $N = 1$,$K = 4$, $B(t) = (1+r)^t$, $r \geq 0$, $S(0) = 5$,

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}$$
$$= 8\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + 4\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega),$$
$$S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2, \omega_3 \\ 3 & \text{if} \quad \omega = \omega_4 \end{cases}$$
$$= 9\mathbf{1}_{\{\omega_1\}}(\omega) + 6\mathbf{1}_{\{\omega_2,\omega_3\}}(\omega) + 3\mathbf{1}_{\{\omega_4\}}(\omega).$$

- Find $Q = (Q_1, Q_2, Q_3, Q_4)^T$ satisfying

$$\mathbb{E}_Q\left[S(t+1)|\mathcal{F}_t\right] = (1+r)S(t), \qquad t = 0, 1.$$

Economic considerations

- There is an alternative way for finding the martingale measure Q.
- This consists in decomposing the multiperiod market in a series of single period markets.
- One then find a risk neutral measure for each of these single period markets.
- The martingale measure for the multiple period market is contructed by “pasting together” these risk neutral measures.

Economic considerations

If Q is a martingale measure and H is a self-financing trading strategy, then $V^* = \{V^*(t)\}_{t=0,\dots,T}$ is a martingale under Q.
[First Fundamental Theorem of Asset Pricing] There do not exist arbitrage opportunities if and only if there exist a martingale measure.

Economic considerations

- All the concepts we saw for single period markets also extend to multiple period markets.

A **linear pricing measure (LPM)** is a non-negative vector $\pi = (\pi_1, ..., \pi_K)^T$ such that for every self-financing trading strategy H you have

$$V^*(0) = \sum_{k=1}^{K} \pi_k V^*(T, \omega_k).$$

- If Q is martingale measure then it is also a **LPM**.
- One can see that any strictly positive **LPM** π must be a martingale measure.

A vector π is a ***LPM*** if an only if π is a probability measure on Ω under which all the discounted price processes are martingales.

Economic considerations

H is a **dominant self-financing trading strategy** if there exists another self-financing trading strategy $\widehat{H}$ such that $V(0) = \widehat{V}(0)$ and $V(T,\omega) > \widehat{V}(T,\omega)$ for all $\omega \in \Omega$. There exists a **LPM** if and only if there are no dominant self-financing trading strategies. We say the the **law of one price** holds for a multiperiod model if there do not exist two self-financing trading strategies, say $\widehat{H}$ and $\widetilde{H}$, such that $\widehat{V}(T,\omega) = \widetilde{V}(T,\omega)$ for all $\omega \in \Omega$ but $\widehat{V}(0) \neq \widetilde{V}(0)$.

- The existence of a linear pricing measure implies that the law of one price hold.

Economic considerations

- Denote

$$W = \left\{ X \in \mathbb{R}^K : X = G^*, \text{ for some self-financing trading strategy } H \right\},$$
$$W^\perp = \left\{ Y \in \mathbb{R}^K : X^T Y = 0, \text{ for all } X \in W \right\},$$
$$A = \left\{ X \in \mathbb{R}^K : X \geq 0, X \neq 0 \right\},$$
$$P = \left\{ X \in \mathbb{R}^K : X_1 + ... + X_K = 1, X \geq 0 \right\},$$
$$P^+ = \{ X \in P : X_1 > 0, ..., X_K > 0 \}.$$

- As with single period markets:
  - We will denote by $\mathbb{M}$ the set of all martingale measures.
  - The set of all linear pricing measures is $P \cap W^\perp$.
  - $\mathbb{M} = P^+ \cap W^\perp$.
  - $W \cap A = \emptyset$ if and only if $\mathbb{M} \neq \emptyset$.
  - $\mathbb{M}$ is convex set whose closure is $P \cap W^\perp$, the set of all linear pricing measures.

Risk Neutral Pricing

Risk neutral pricing

A **contingent claim** is a random variable X representing the payoff at time T of a financial contract which depends on the values of the risky assets in the market.

Example

Consider the market with $T = 2$, $K = 4$, $S(0) = 5$,

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}, \; S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2, \omega_3 \\ 3 & \text{if} \quad \omega = \omega_4 \end{cases}.$$

- $X = (S(2) - 5)^+$. **European call option** with strike 5.

$$X = (\max(0, 9-5), \max(0, 6-5), \max(0, 6-5), \max(0, 3-5))^T = (4, 1, 1, 0)^T.$$

Risk neutral pricing

Example 3

- $Y = \left(\frac{1}{3}\sum_{t=0}^{2} S(t) - 5\right)^{+}$. **Asian call option** with strike 5.

$$Y_1 = \left(\frac{1}{3}\sum_{t=0}^{2} S(t,\omega_1) - 5\right)^{+} = \max\left(0, \frac{1}{3}(5+8+9) - 5\right) = 7/3,$$

$$Y_2 = \left(\frac{1}{3}\sum_{t=0}^{2} S(t,\omega_2) - 5\right)^{+} = \max\left(0, \frac{1}{3}(5+8+6) - 5\right) = 4/3,$$

$$Y_3 = \left(\frac{1}{3}\sum_{t=0}^{2} S(t,\omega_3) - 5\right)^{+} = \max\left(0, \frac{1}{3}(5+4+6) - 5\right) = 0,$$

$$Y_4 = \left(\frac{1}{3}\sum_{t=0}^{2} S(t,\omega_4) - 5\right)^{+} = \max\left(0, \frac{1}{3}(5+4+3) - 5\right) = 0,$$

which yields $Y = (7/3, 4/3, 0, 0)^T$.

Risk neutral pricing

Standing Assumption: The financial market model is arbitrage free. A contingent claim X is **attainable** (or **marketable**) if there exists a self-financing trading strategy such that $V(T) = X$.
Such strategy is said to replicate or generate or hedge X. The time t value of an attainable contingent claim X, denoted by $P_X(t)$,is equal to $V(t)$, the time t value of a portfolio generating X. Moreover,

$$V(t) = \mathbb{E}_Q\left[\left.\frac{B(t)}{B(T)}X\right|\mathcal{F}_t\right], \qquad t = 0, ..., T, \quad Q \in \mathbb{M}.$$

Risk neutral pricing

- In order to sell a contingent claim X the seller must find the trading strategy that replicates/hedges X.
- We will see three methods for finding a hedging strategy.

First method

- We must know the value process $V = \{V(t)\}_{t=0,\dots,T}$.
- We solve

$$V(t) = H_0(t) + \sum_{n=1}^{N} H_n(t) S_n(t), \qquad t = 1, \dots, T,$$

taking into account that H must be predictable.

Risk neutral pricing

Second method

- All we know is X.
- In this method, we work backwards in time and find $V(t)$ and $H(t)$ simultaneously.
- Since $V(T) = X$, we first find $H(T)$ by taking into account that H is predictable and solving

$$X = H_0(T)\, B(T) + \sum_{n=1}^{N} H_n(T)\, S_n(T) \,.$$

- Using that H is must be self-financing, we find $V(T-1)$ by computing

$$V(T-1) = H_0(T)\, B(T-1) + \sum_{n=1}^{N} H_n(T)\, S_n(T-1) \,.$$

Risk neutral pricing

Second method

- Next, taking into account that H is predictable, we find $H\left(T-1\right)$ by solving

$$V\left(T-1\right)=H_0\left(T-1\right)B\left(T-1\right)+\sum_{n=1}^{N}H_n\left(T-1\right)S_n\left(T-1\right).$$

- We repeat this procedure until computing $V\left(0\right)$.

Risk neutral pricing

Third method

- It relies on the fact that the self-financing condition

$$V^*(0) + G^*(t) = V^*(t),$$

is equivalent to

$$V^*(t-1) + \sum_{n=1}^{N} H_n(t)\,\Delta S_n^*(t) = V^*(t).$$

- We can use this system of equations, together with the predictability condition on $H(t) = (H_1(t), ..., H_N(t))^T$, to find $V^*(t-1)$ and $H(t)$.

Risk neutral pricing

Third method

- Then, we can find

$$H_0(t) = V^*(t-1) - \sum_{n=1}^{N} H_n(t) S_n^*(t-1),$$
$$V(t-1) = B(t-1) V^*(t-1).$$

- We begin with $V^*(T) = X/B(T)$ and work backwards in time.

Risk neutral pricing

Example (Continuation Example 3)

Consider the market with $T = 2$, $K = 4$, $S(0) = 5$,

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}, \; S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2, \omega_3 \\ 3 & \text{if} \quad \omega = \omega_4 \end{cases}.$$

- Suppose $r = 0$. We know that $Q = (1/6, 1/12, 1/4, 1/2)^T$ is the unique martingale measure in this market.
- Consider $X = (S(2) - 5)^+$ and $Y = \left(\frac{1}{3}\sum_{t=0}^{2} S(t) - 5\right)^+$ or in vector notation $X = (4, 1, 1, 0)^T$ and $Y = (7/3, 4/3, 0, 0)^T$.
- Compute the price of X for each t and a self-financing trading strategy generating X. (Using the first and second methods)
- Do the same for Y using the third method.

Complete and Incomplete Markets

Complete and incomplete markets

A market is **complete** if every contingent claim X is attainable. Otherwise, it is called **incomplete**. A multiperiod market is complete if and only if every underlying single period market is complete.

The backward procedures explained in the last section work if and only every underlying single period market is complete.

The criterion given in Proposition 35, in general, is not a practical characterization of market completeness.

Complete and incomplete markets

Theorem

A multiperiod market is complete if and only if $\exists$ RNPM.

Proposition

A contingent claim X is attainable if and only if $\mathbb{E}_Q\left[X/B\left(T\right)\right]$ takes the same value for every Q.

Complete and incomplete markets

Example

- Consider the market with $K = 5, T = 2, r = 0, S(0) = 5,$

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2, \omega_3 \\ 4 & \text{if} \quad \omega = \omega_4, \omega_5 \end{cases},$$

$$S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 7 & \text{if} \quad \omega = \omega_2 \\ 6 & \text{if} \quad \omega = \omega_3, \omega_4 \\ 3 & \text{if} \quad \omega = \omega_5 \end{cases}.$$

- One can check (exercise) that

$$Q_\lambda = \left(\frac{\lambda}{4}, \frac{(2-3\lambda)}{4}, \frac{(2\lambda-1)}{4}, \frac{1}{4}, \frac{1}{2} \right)^T, \frac{1}{2} < \lambda < \frac{2}{3}.$$

Complete and incomplete markets

Example 5

- A contingent claim $X = (X_1, X_2, X_3, X_4, X_5)^T$ is attainable if and only if

$$\begin{aligned}\mathbb{E}_Q\left[\frac{X}{B(2)}\right] &= \mathbb{E}_Q[X] \\ &= X_1\frac{\lambda}{4} + X_2\frac{(2-3\lambda)}{4} + X_3\frac{(2\lambda-1)}{4} + X_4\frac{1}{4} + X_5\frac{1}{2} \\ &= \frac{\lambda}{4}(X_1 - 3X_2 + 2X_3) + \frac{1}{4}(2X_2 - X_3 + X_4 + 2X_5),\end{aligned}$$

  does not depend on λ.
- Hence, we can conclude that X is attainable if and only if

$$X_1 - 3X_2 + 2X_3 = 0.$$

Optimal Portfolio Problem

Optimal portfolio problem

- Let U be an utility function as in section 5.1.
- We are interested in the following optimization problem:

$$\left.\begin{array}{cc} \max & \mathbb{E}\left[U\left(V\left(T\right)\right)\right] \\ \text{subject to} & V\left(0\right)=v, \\ & H\in\mathcal{H}, \end{array}\right\} \tag{12}$$

  where $v\in\mathbb{R}$ and $\mathcal{H}:=\{\text{set of all self-financing trading strategies}\}$.
- Recall that $V\left(T\right)=V^{*}\left(T\right)B\left(T\right)$, $V^{*}\left(T\right)=V^{*}\left(0\right)+G^{*}\left(T\right)$. Therefore, (12) is equivalent to

$$\left.\begin{array}{cc} \max & \mathbb{E}\left[U\left(B\left(T\right)\left\{v+G^{*}\left(T\right)\right\}\right)\right] \\ \text{subject to} & H=\left(H_{1},...,H_{N}\right)^{T}\in\mathcal{H}_{P}, \end{array}\right\} \tag{13}$$

  where $v\in\mathbb{R}$ and
  $\mathcal{H}_{P}:=\left\{\text{set of all predictable processes taking values in }\mathbb{R}^{N}\right\}$.
- If $(\widehat{H}_{1},...,\widehat{H}_{N})^{T}$ is a solution of (13), then one can find $\widehat{H}_{0}$ such that $\widehat{H}=(\widehat{H}_{0},\widehat{H}_{1},...,\widehat{H}_{N})^{T}$ is self-financing and $V\left(0\right)=v$, giving a solution to (12).

Optimal portfolio problem

Proposition

If H is a solution of (12) and V is its associated porfolio value process then

$$Q\left(\omega\right)=\frac{B\left(T,\omega\right)U'\left(V\left(T,\omega\right),\omega\right)}{\mathbb{E}\left[B\left(T\right)U'\left(V\left(T\right)\right)\right]}P\left(\omega\right),\qquad \omega\in\Omega,$$

is a martingale measure.

Optimal portfolio problem

- There are several methods to solve the optimal portfolio problem:
  - Direct approach (classical optimization problem taking into account predictability)
  - Dynamic programming.
  - Martingale method.
- We will only consider the martingale method in these lectures.
- This method is analogous to the risk neutral computational approach in single period financial markets.
- We will assume that:
  - The market is arbitrage free and complete: $\mathbb{M} = \{Q\}$.
  - U does not depend on ω.
- The martingale method can be split in 3 steps.

Optimal portfolio problem

Step 1

- Identify the set W_v of attainable wealths:

$$W_v = \left\{ W \in \mathbb{R}^K : W = V(T) \text{ for some } H \in \mathcal{H} \text{ with } V(0) = v \right\}.$$

- If the model is complete

$$W_v = \left\{ W \in \mathbb{R}^K : \mathbb{E}_Q \left[W / B(T) \right] = v \right\}.$$

Optimal portfolio problem

Step 2

- We need to solve the problem

$$\left.\begin{array}{cc} \max & \mathbb{E}\left[U\left(W\right)\right] \\ \text{subject to} & W \in W_v, \end{array}\right\} \tag{14}$$

- To solve (14) we will use the method of Lagrange multipliers.
- Consider the Lagrange function

$$\begin{aligned}\mathcal{L}\left(W;\lambda\right) &= \mathbb{E}\left[U\left(W\right)\right] - \lambda\left(\mathbb{E}_Q\left[W/B\left(T\right)\right] - v\right) \\ &= \mathbb{E}\left[U\left(W\right)\right] - \lambda\mathbb{E}_Q\left[W/B\left(T\right) - v\right] \\ &= \mathbb{E}\left[U\left(W\right)\right] - \lambda\mathbb{E}\left[L\left(W/B\left(T\right) - v\right)\right] \\ &= \mathbb{E}\left[U\left(W\right) - \lambda L\left(\frac{W}{B\left(T\right)} - v\right)\right].\end{aligned}$$

Optimal portfolio problem

Step 2

- The first optimality condition gives

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda}(W;\lambda) = \mathbb{E}_Q\left[W/B(T)\right] - v$$
$$0 = \frac{\partial \mathcal{L}}{\partial W_k}(W;\lambda) = P(\omega_k)\left\{U'(W(\omega_k)) - \lambda \frac{L(\omega_k)}{B(T,\omega_k)}\right\},$$

  for $k = 1, ..., K.$

- Then the optimum $(\widehat{\lambda}, \widehat{W})$ satisfies

$$\mathbb{E}_Q\left[\widehat{W}/B(T)\right] = v, \qquad U'\left(\widehat{W}\right) = \widehat{\lambda}\frac{L}{B(T)}.$$

Optimal portfolio problem

Step 2

- To solve these equations, we consider $I(y) := (U')^{-1}(y)$ and compute $\widehat{W} = I\left(\widehat{\lambda}\frac{L}{B(T)}\right)$, then $\widehat{\lambda}$ is chosen such that

$$\mathbb{E}_Q\left[I\left(\widehat{\lambda}LB^{-1}(T)\right)B^{-1}(T)\right] = v,$$

holds.

Step 3

- Given the optimal wealth $\widehat{W}$, find a self-financing trading strategy $\widehat{H}$ that generates $\widehat{W}$.
- We use the second method for findind a replicating strategy.

Optimal portfolio problem

Example

- Consider the market with $T = 2$, $K = 4$, $S(0) = 5$,

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases},$$

$$S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2, \omega_3 \\ 3 & \text{if} \quad \omega = \omega_4 \end{cases},$$

  $0 \leq r < 1/8$ and $P = (1/4, 1/4, 1/4, 1/4)^T$.
- We want to solve the optimal portfolio problem with $U(u) = \log(u)$.
- We will discuss this example on the smartboard.

Thank you!