Models of changes in the value of money

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Contents

Introduction

Introduction

- It is common sense that 1000 NOK to be received after one year are worth less than the same amount today.
- The main reason is that money due in the future or locked in a fixed term account cannot be spent right away.
- Therefore, one would expect to be compensated for the postponed consumption.
- In addition, prices of goods may rise in the meantime and the amount will not have the same purchasing power as it would have at present.

We will be concerned with two questions:

- What is the ***future value*** of an amount invested or borrowed today?
- What is the ***present value*** of an amount to be paid or received at a certain time in the future?

Simple Interest

Simple Interest

- In exchange for the use of a depositor's money, banks pay a fraction of the account balance back to the depositor. This fractional payment is known as ***interest***.
- Alternatively, it is the fee charged to you when borrowing money from a bank.
- Suppose that an amount is deposited into a bank account. This initial deposit is called the ***principal*** and it is denoted by P.
- We shall first consider the case when interest is attracted only by the principal. This is called ***simple interest***.

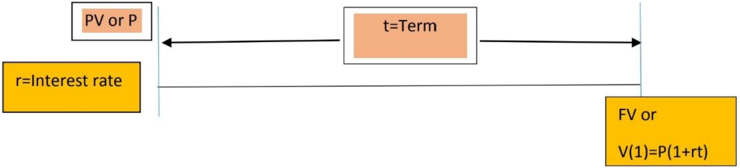
Simple Interest

We shall first consider the case when interest is attracted only by the principal. This is called ***simple interest***.
After one year the interest earned will be rP, where $r > 0$ is the ***interest rate***.
The value of the investment will become

$$V(1) = \text{Principal Value} + \text{Interest};$$

$$V(1) = P + I;$$

$$V(1) = P + rP = (1 + r)P.$$



Simple Interest

$$V(1) = P + rP = (1 + r)P.$$

After two years, the investment will grow to

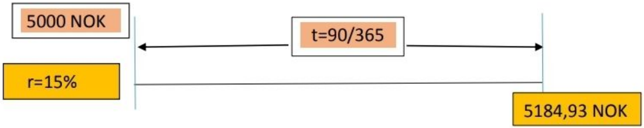
$$V(2) = P + rP + rP = (1 + 2r)P.$$

Consider a fraction of a year. Interest is typically calculated on a daily basis: the interest earned in one day will be $\frac{1}{365}rP$. After n days the total value of the investment will become

$$V\left(\frac{n}{365}\right) = P + \frac{n}{365}rP = \left(1 + \frac{n}{365}r\right)P.$$

Example

Calculate the simple interest and sum accumulated for 5000 NOK put on the bank acccount for 90 days at 15 % per annum.



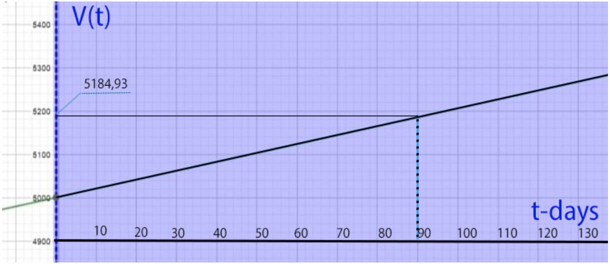


Figure: Principal attracting simple interest at $r = 0.15, P = 5000$.

Simple interest

Rules of simple interest

- The future value of the principal at time t, denoted by $V(t)$, is given by

$$V(t) = (1 + rt)P, \tag{1}$$

where time t, expressed in years, is any nonnegative real number.

- The number $1 + rt$ is called the ***growth factor*** and it is linear in t.
- If the principal P is invested at time s (rather than 0), then the value of the investment at time $t \geq s$ will be

$$V(t) = (1 + (t - s)r)P$$

Simple interest

Remark

In the case of a time-varying interest rate, interest is calculated separately for each period during which the interest rate has changed. Capital accumulated over time $t = \sum_{i=1}^{k} t_i$ is calculated according to the formula

$$V(t) = \left(1 + \sum_{i=1}^{k} t_i r_i\right) P,$$

where
i is the number of interest rate change periods,
t_i is the duration of the period,
r_i is the interest rate in the period .

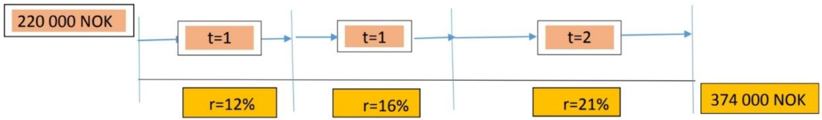
Simple interest

Example

A credit in the amount of 220,000 NOK was granted for 4 years: Interest rates 1-st year - 12 % , 2-d year - 16 % , 3-d and 4-th years - 21 % .
Find the accumulated amount to repay the debt using the simple interest method.

Solution

$$V(4) = P(1 + r_1 \cdot t_1 + r_2 \cdot t_2 + r_3 \cdot t_3) =$$
$$= 220000(1 + 0,12 + 0,16 + 0,21 \cdot 2) = 374000$$



Simple interest

- The ***return*** of an investment starting at time s and finishing at time t will be denoted by $K(s,t)$ and is defined by

$$K(s,t) := \frac{V(t) - V(s)}{V(s)}. \tag{2}$$

- In the case of simple interest $K(s,t) = (t-s)\,r$.
- In particular, the interest rate is equal to the rate of return over one year, $K(s,t) = r$, where s and t are such that $|t-s| = 1$.
- $K(0,t) = K(0,s) + K(s,t)$ for $s < t$.

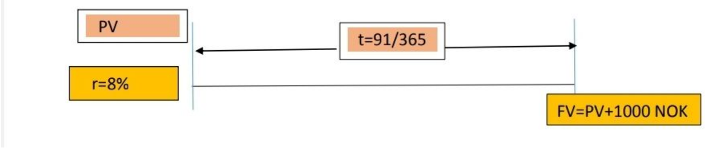
Rule

Interest rates will always refer to a period of one year, facilitating the comparison between different investments, independently of their actual duration. By contrast, the return reflects both the interest rate and the length of time the investment is held.

Simple interest

Exercise

Find the principal to be deposited initially in an account attracting simple interest at a rate of 8 % if 1 000 NOK is needed after three months (91 days)



Problem is to find the initial amount whose value at time t is given. In the case of simple interest the answer is obtained by solving equation (1) for the principal P obtaining

$$V(0) = PV = P = (1 + rt)^{-1} V(t). \tag{3}$$

This quantity is called the ***present or discounted value*** of $V(t)$ and $(1 + rt)^{-1}$ is called the ***discount factor***.

The time value of money

The time value of money

Definition

A particular investment has different values on different dates.

1 000 NOK today will not be the same as 1 000 NOK in six-months' time. In fact, if the prevailing simple interest rate is 16 % per annum, then, in six months, the 1 000 NOK will have accumulated to 1 080 NOK.

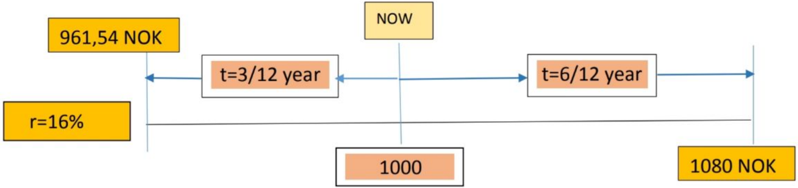
$$1000 \cdot (1 + \frac{1}{12} \cdot 0,16) = 1080$$

On the other hand, three months ago it was worth less –to be precise, it was worth 961,54 NOK.

$$\frac{1000}{\left(1 + \frac{1}{4} \cdot 0,16\right)} = 961,54$$

The time value of money

Represented on a time line, these statements yield the following picture:



Rules

- To move money *forward* (determine a future value) where simple interest is applicable, inflate the relevant sum by *multiplying* by the factor $(1 + rt)$.
- To move money *backward* (determine a present value) where simple interest is applicable, deflate the relevant sum by *dividing* by the factor $(1 + rt)$.

Discrete or Periodic Compounding

Discrete or Periodic Compounding

Suppose that an amount P is deposited in a bank account earning interest at a constant rate $r > 0$.

However, now we assume that the interest earned will be added to the principal periodically (for example: annually, semi-annually, quarterly, monthly, daily).

Subsequently, interest will be earned by the principal and all the interest earned so far.

In this case, we talk of ***discrete or periodic compounding***.

Discrete or Periodic Compounding

If m interest payments are made per year, the time between two consecutive payments measured in years will be $1/m$, the first interest payment due at time $1/m$.

Each interest payment will increase the principal by a factor of $1 + \frac{r}{m}$.

Hence, the ***future value*** of the principal will become

$$V(t) = \left(1 + \frac{r}{m}\right)^{tm} P, \tag{4}$$

because there will be tm interest payments during this period.

The amount $\left(1 + \frac{r}{m}\right)^{tm}$ is also called the ***growth factor***.
Note that, in this case, the growth factor is not a linear function of time.

Discrete or periodic compounding

Example

Find the amount to which 1500 NOK will grow if compounded quarterly at 1% for 5 years.
We have $m = 4$, $r = 0.01$ and $t = 5$, which yields

$$V(5) = \left(1 + \frac{0.01}{4}\right)^{5 \times 4} \times 1500 \simeq 1576.81.$$

PROPOSITION

The future value $V(t)$ increases if one of the parameters m, t, r or P increases, the others remaining unchanged.

Discrete or periodic compounding

- In this case the formula for the ***present or discounted value*** of $V(t)$ is given by

$$V(0) = V(t)\left(1 + \frac{r}{m}\right)^{-tm}, \tag{5}$$

where $\left(1 + \frac{r}{m}\right)^{-tm}$ is the ***discount factor***.

- The value of an investment at time $0 < t < T$, given the value $V(T)$, assuming periodic compounding with frequency m and interest rate r, is given by

$$V(t) = \left(1 + \frac{r}{m}\right)^{-(T-t)m} V(T). \tag{6}$$

- The ***rate of return*** on a deposit attracting interest compounded periodically is given by

$$K(s,t) = \frac{V(t) - V(s)}{V(s)} = \left(1 + \frac{r}{m}\right)^{(t-s)m} - 1. \tag{7}$$

Discrete or periodic compounding

Remark

The rate return on a deposit subject to periodic compounding is not additive. Take $m = 1$, then

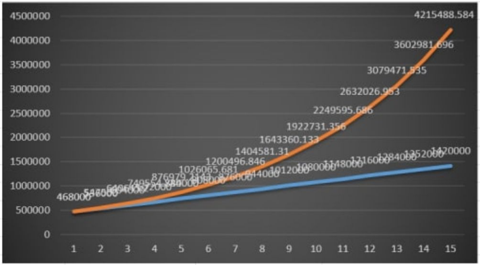
$$K(0,1) = K(1,2) = r,$$
$$K(0,2) = (1+r)^2 - 1 = 2r + r^2,$$

and clearly these equations imply that

$$K(0,1) + K(1,2) \neq K(0,2).$$

Comparison of Simple Interest and Periodic Compounding

Let $V(0) = 400000$ NOK, $r = 17\%$.



Simple interest		Periodic Compounding	
Year	V(t)	Year	V(t)
1	468000	1	468000
2	536000	2	547560
3	604000	3	640645.2
4	672000	4	749554.884
5	740000	5	876979.2143
6	808000	6	1026065.681
7	876000	7	1200496.846
8	944000	8	1404581.31
9	1012000	9	1643360.133
10	1080000	10	1922731.356
11	1148000	11	2249595.686
12	1216000	12	2632026.953
13	1284000	13	3079471.535
14	1352000	14	3602981.696
15	1420000	15	4215488.584

Sheet1

Continuous Compounding

Continuous compounding

- ***Continuous compounding*** is obtained by considering periodic compounding and increasing to infinity the compounding frequency.
- That is, the ***future value*** of a pricincipal P under continuous compounding is

$$\begin{aligned} V(t) &= \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{tm} P \\ &= \lim_{m\to\infty}\left(1+\frac{rt}{tm}\right)^{tm} P \\ &= e^{rt} P. \end{aligned} \tag{8}$$

Continuous compounding

- The corresponding ***growth factor*** is e^{rt}.
- Formula (8) is a good approximation of the case of periodic compounding when m is large.
- Continuous compounding is simpler and it is the method of compounding used in continuous time models (for example, in the Black-Scholes model).

Continuous compounding

Proposition

Continuous compounding produces higher future value than periodic compounding with any frequency m (same principal P and interest rate r).

Proof.

Fix $t > 0$. By Proposition (21) we get that

$$\left(1 + \frac{r}{m}\right)^{tm} < \left(1 + \frac{r}{k}\right)^{tk}, \qquad m < k.$$

Hence, the sequence $\left\{\left(1 + \frac{r}{m}\right)^{tm}\right\}_{m \geq 1}$ is monotonically increasing and converges from below to e^{rt}. □

Continuous compounding

- The ***present value*** under continuous compounding is given by

$$V(0) = V(t)\, e^{-rt}$$

.

- The ***discount factor*** is e^{-rt}.
- Given the terminal value $V(T)$, we have

$$V(t) = V(T)\, e^{-r(T-t)}, \qquad 0 \leq t \leq T. \tag{9}$$

- The ***rate of return***

$$K(s,t) = \frac{V(t) - V(s)}{V(s)} = \frac{e^{r(t-s)}V(s) - V(s)}{V(s)} = e^{r(t-s)} - 1,\ \ s < t.$$

The rate of return $K(s,t)$ also fails to be additive.

Continuous compounding

- In this case is convenient to introduce the ***logarithmic rate of return***, defined by

$$k(s,t) := \log\left(\frac{V(t)}{V(s)}\right). \tag{10}$$

The following result is trivial. The logarithmic rate of return is additive, that is,

$$k(s,t) + k(t,u) = k(s,u), \tag{11}$$

for $0 \leq s \leq t \leq u$.

Example

Example

Find the present value of $1,000,000$ NOK to be received after 20 years assuming continuous compounding at 6 % .

The present value under continuous compounding is given by

$$V(0) = V(t)\, e^{-rt} = 1000000 \cdot e^{-0{,}06 \cdot 20} = 301194.$$

Comparing Compounding Methods

Comparing compounding methods

- The idea is to compare growth factors over a fixed period of time, usually one year.

Definition

We say that two compounding methods are ***equivalent*** if the corresponding growth factors over a period of one year are the same. If one of the growth factors exceeds the other, then the corresponding compounding method is said to be ***preferable***.

Example

Semi-annual compounding at 10% is equivalent to annual compounding at 10.25%. Indeed,

$$\left(1+\frac{0.1}{2}\right)^2 = 1.1025 \qquad \text{(Semi-annual)}$$

$$\left(1+\frac{0.1025}{1}\right)^1 = 1.1025 \qquad \text{(Annual)}$$

Comparing compounding methods

- One can switch from one compounding method to other compounding method by changing the interest rate.
- We shall normally use either annual or continuous compounding.

Definition

For a given compounding method with interest rate r, the ***effective (annual) rate*** r_e is the one that gives the same growth factor over a one year period under annual compounding.

In particular, in the case of periodic compounding with frequency m and rate r the effective rate satisfies

$$\left(1+\frac{r}{m}\right)^m = 1 + r_e. \tag{12}$$

In the case of continuous compounding with rate r we have

$$e^r = 1 + r_e. \tag{13}$$

Comparing compounding methods

Two compounding methods are equivalent if and only if the corresponding effective rate r_e and r'_e are equal $r_e = r'_e$.

- In terms of the effective rate r_e the future value can be written as

$$V(t) = (1 + r_e)^t P, \qquad t \geq 0.$$

- This applies to both, continuous compounding and periodic compounding.
- Simple interest does not fit into the scheme for comparing compounding methods.
- In this case, the future value $V(t)$ is a linear function of time t, whereas it is an exponential if either continuous or periodic compounding applies.

Money Market

Money market

- The money market consists of risk free securities, typically bonds.

Bonds are financial assets promising the holder a sequence of guaranteed future payments.

- Risk free means here that these payments will be made with certainty.

A ***zero-coupon bond*** is a bond involving just a single future payment.

- The issuing institution of a zero-coupon bond promises to exchange the bond for a certain amount of money F, the ***face value***, on a given day T, the ***maturity date***.

Money market

- Effectively, the person who buys the bond is lending money to the bond issuer/writer.
- Given an interest rate r and a maturity date, say one year, the present value of the bond should be

$$V(0) = F(1+r)^{-1}. \tag{14}$$

- In reality, the opposite happens: bonds are freely traded and their prices are driven by market forces, whereas the interest rate is implied by the bond prices,

$$r = \frac{F}{V(0)} - 1. \tag{15}$$

- This formula gives the ***implied annual compounding rate***.

Money market

- We will assume from now on that $F = 1$.
- Bonds can be sold at any time t prior to maturity time T at some price, denoted by $B(t,T)$.
- Note that $B(T,T) = F = 1$.
- Again these prices determine the implied interest rates.
- By applying formulas (6) and (9) with $V(t) = B(t,T)$ and $V(T) = 1$, we get

$$B(t,T) = \left(1 + \frac{r_m}{m}\right)^{-m(T-t)}, \tag{16}$$

and

$$B(t,T) = e^{-r_c(T-t)}. \tag{17}$$

Money market

- These different implied rates are equivalent, since the bond price does not depend on the compounding method used.
- In general, the implied interest rate may depend on the trading time as well as the maturity date T.
- The dependence on T is called the ***term structure***.
- Actually, the most advanced models for interest rates model $B\left(t, T\right)$ as a two parameter stochastic process.
- In this course we will adopt the simplifying assumption that the interest rate remains constant.

Bank Account

Bank account

- Investment banks offer the possibility of investing in the money market by buying and selling bonds on behalf of its customers.
- Actually, when you open a bank account attracting some interest r, the bank trades in the money market to pay you that interest.
- Suppose you open a deposit for T years and you set an initial amount $A(0)$.
- With $A(0)$ the bank buys $A(0)/B(0,T)$ bonds.
- The value of each bond if sold at time $t < T$ yields

$$B(t,T) = e^{-r(T-t)} = e^{rt}e^{-rT} = e^{rt}B(0,T).$$

Bank account

- As a result, the investment done by the bank gives

$$A(t) = \frac{A(0)}{B(0,T)} B(t,T) = e^{rt} A(0), \qquad t \leq T, \tag{18}$$

which is also the value of your bank account.

- One can extend this procedure for all t by buying new bonds with longer maturities.

Thank you!