Risk-free financial assets

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Money Market

Money market

- The money market consists of risk free securities, typically bonds.

Bonds are financial assets promising the holder a sequence of guaranteed future payments.

- Risk free means here that these payments will be made with certainty.

A ***zero-coupon bond*** is a bond involving just a single future payment.

- The issuing institution of a zero-coupon bond promises to exchange the bond for a certain amount of money F, the ***face value***, on a given day T, the ***maturity date***.

Money market

- Effectively, the person who buys the bond is lending money to the bond issuer/writer.
- Given an interest rate r and a maturity date, say one year, the present value of the bond should be

$$V(0) = F(1+r)^{-1}. \tag{1}$$

- In reality, the opposite happens: bonds are freely traded and their prices are driven by market forces, whereas the interest rate is implied by the bond prices,

$$r = \frac{F}{V(0)} - 1. \tag{2}$$

- This formula gives the ***implied annual compounding rate***.

Money market

- We will assume from now on that $F = 1$.
- Bonds can be sold at any time t prior to maturity time T at some price, denoted by $B(t, T)$.
- Note that $B(T, T) = F = 1$.
- Again these prices determine the implied interest rates.
- By applying formulas $V(t) = V(T)\left(1 + \frac{r}{m}\right)^{-(T-t)m}$ and $V(t) = e^{-r(T-t)}V(T)$ with $V(t) = B(t, T)$ and $V(T) = 1$, we get

$$B(t, T) = \left(1 + \frac{r_m}{m}\right)^{-m(T-t)}, \tag{3}$$

and

$$B(t, T) = e^{-r_c(T-t)}. \tag{4}$$

Money market

- These different implied rates are equivalent, since the bond price does not depend on the compounding method used.
- In general, the implied interest rate may depend on the trading time as well as the maturity date T.
- The dependence on T is called the ***term structure***.
- Actually, the most advanced models for interest rates model $B(t, T)$ as a two parameter stochastic process.
- In this course we will adopt the simplifying assumption that the interest rate remains constant.

Money market

Example

An investor paid 95000 NOK for a bond with face value 100000 NOK maturing in six months. When will the bond value reach 99000 NOK if the interest rate remains constant?

Solution. For given $F = 100000$ NOK and $V(0) = 95000$ NOK we solve the equation

$$100000 = 95000 \cdot \left(1 + \frac{6}{12}r\right)^{\frac{6}{12}},$$

or

$$1,0526315789 = (1 + 0.5r)^{0.5}$$

for r to find the implied effective rate to be about 10.80 %. If this rate remains constant, then the bond price will reach 99000 NOK at a time t such that

$$100000 = 99000(1 + r)^{(0.5-t)}.$$

The solution is $t \approx 0.402$ years, that is, about $0.402 \cdot 365 \approx 146.73$ days. The bond price will reach 99000 NOK on day 147.

Money market

Bonds promising a sequence of payments are called ***coupon bonds***. These payments consist of the face value due at maturity, and coupons paid regularly, typically annually, semi-annually, or quarterly, the last coupon due at maturity. The assumption of constant interest rates allows us to compute the price of a coupon bond by discounting all the future payments.

Example

Consider a bond with face value $F = 100000$ NOK maturing in five years, $T = 5$, with coupons of $C = 100$ NOK paid annually, the last one at maturity. This means a stream of payments of $100, 100, 100, 100, 1110$ NOK at the end of each consecutive year. Given the continuous compounding rate r, say 12 %, we can find the price of the bond:

$$V(0) = 100e^{-r} + 100e^{-2r} + 100e^{-3r} + 100e^{-4r} + 1100e^{-5r} =$$

$$= 100e^{-0.12} + 100e^{-0.24r} + 100e^{-0.36} + 100e^{-0.48} + 1100e^{-0.6} \approx 902.69.$$

Money market

The coupon can be expressed as a fraction of the face value. Assuming that coupons are paid annually, we shall write $C = iF$, where i is called the coupon rate.

Proposition

Whenever coupons are paid annually, the coupon rate is equal to the interest rate for annual compounding if and only if the price of the bond is equal to its face value. In this case we say that the bond sells at par.

- If a bond sells below the face value, it means that the implied interest rate is higher than the coupon rate (since the price of a bond decreases when the interest rate goes up). If the bond price is higher than the face value, it means that the interest rate is lower than the coupon rate. This may be important information in real circumstances, where the bond price is determined by the market and gives an indication of the level of interest rates.

Money market

Price of the bond is the discounted value of cash flows

- Step 1: Determine cash flows from bond characteristics
- Step 2: Discount cash flows

Coupon 5% bond
Face Value 1000
Interest rate 5%

Today	C	Future Value	Price
			1000
$t=1$	50		47.62
$t=2$	50		45.35
$t=3$	50		43.19
$t=4$	50		41.14
$t=5$	50	1000	822.70

Coupon 4% bond
Face Value 1000
Interest rate 3.29%

Today	C	Future Value	Price
			1032.2
$t=1$	40		38.7
$t=2$	40		37.5
$t=3$	40		36.3
$t=4$	40		35.1
$t=5$	40	1000	884.6

$$\text{Price} = \sum_{k=1}^{n-1} \frac{C}{(1+r)^k} + \frac{C+F}{(1+r)^n}$$

Bank Account

Bank account

- Investment banks offer the possibility of investing in the money market by buying and selling bonds on behalf of its customers.
- Actually, when you open a bank account attracting some interest r, the bank trades in the money market to pay you that interest.
- Suppose you open a deposit for T years and you set an initial amount $A(0)$.
- With $A(0)$ the bank buys $\frac{A(0)}{B(0,T)}$ bonds.
- The value of each bond if sold at time $t < T$ yields

$$B(t,T) = e^{-r(T-t)} = e^{rt}e^{-rT} = e^{rt}B(0,T).$$

Bank account

- As a result, the investment done by the bank gives

$$A(t) = \frac{A(0)}{B(0,T)} B(t,T) = e^{rt} A(0), \qquad t \leq T, \tag{5}$$

which is also the value of your bank account.

- One can extend this procedure for all t by buying new bonds with longer maturities.

Bank account

The investment in a bond has a finite time horizon. It will be terminated with

$$A(T) = A(0)e^{rT}$$

at the time T of maturity of the bond.
To extend the position in the money market beyond T one can reinvest the amount $A(T)$ into a bond newly issued at time T, maturing at $T' > T$. Taking $A(T)$as the initial investment with T playing the role of the starting time, we have

$$A(t') = A(T)e^{(t'-T)} = A(0)e^{rt'} \text{ for } T < t' < T'.$$

By repeating this argument, we readily arrive at the conclusion that an investment in the money market can be prolonged for as long as required, the formula being valid for all $t \geq 0$.

Bank account

Example

Suppose that one dollar is invested in zero-coupon bonds maturing after one year. At the end of each year the proceeds are reinvested in new bonds of the same kind. How many bonds will be purchased at the end of year 9? Express the answer in terms of the implied continuous compounding rate.

- At time $t = 0$ we buy
$$\frac{1}{B(0,1)} = e^r \text{ bonds;}$$
- at time $t = 1$ we increase our holdings to
$$\frac{e^r}{B(1,2)} = e^{2r} \text{ bonds;}$$
- ...
- at time $t = n$ we purchase
$$\frac{e^r}{B(n,n+1)} = e^{(n+1)r}, \text{ one-year bonds.}$$

Bank account

An alternative way to prolong an investment in the money market for as long as required is to reinvest the face value of any bonds maturing at time T in other bonds issued at time 0, but maturing at a later time $t > T$. Having invested $A(0)$ initially to buy unit bonds maturing at time T , we will have the sum of $\frac{A(0)}{B(0,T)}$ at our disposal at time T.
At this time we chose a bond maturing at time t, its price at T being $B(T,t)$.
At time t this investment will be worth

$$\frac{A(0)}{B(0,T)B(T,t)} = \frac{A(0)}{B(0,t)} = A(0)e^{rt}.$$

Bank account

Finally, consider coupon bonds as a tool to manufacture an investment in the money market. Suppose for simplicity that the first coupon C is due after one year. At time $t = 0$ we buy $\frac{A(0)}{V(0)}$ coupon bonds. After one year we cash the coupon and sell the bond for $V(1)$, receiving the total sum

$$C + V(1) = V(0)e^{r}.$$

Because the interest rate is constant, this sum of money is certain. In this way we have effectively created a zero-coupon bond with face value $V(0)e^{r}$ maturing at time $t = 1$. It means that the scheme worked out above for zero-coupon bonds applies to coupon bonds.

Thank you!