Risk financial assets

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Oslo 2022.9.7

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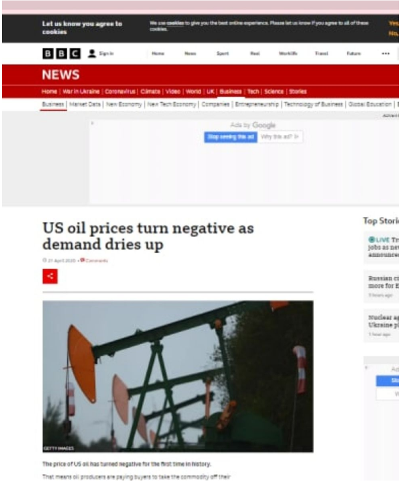
- The price of stock at time t will be denoted by $S(t)$. It is assumed to be strictly positive for all t. We take $t = 0$ to be the present time, $S(0)$ being the current stock price, known to all investors.
- The future prices $S(t)$ for$t > 0$ remain unknown, in general.
- Mathematically, $S(t)$ can be represented as a positive random variable on a probability space Ω, that is,

$$S(t) : \Omega \to (0, \infty).$$

The probability space Ω consists of all feasible price movement 'scenarios' $\omega \in \Omega$. We shall write $S(t, \omega)$ to denote the price at time t if the market follows scenario $\omega \in \Omega$.

Remark

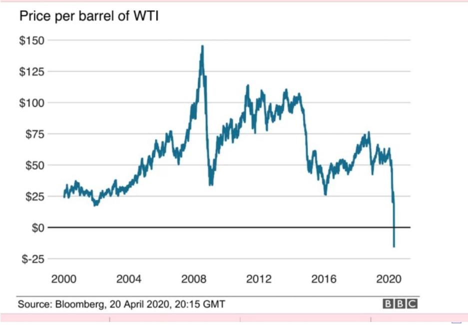
https://www.bbc.com/news/business-52350082



Remark

The price of US oil has turned negative for the first time in history.

That means oil producers are paying buyers to take the commodity off their hands over fears that storage capacity could run out in May.
Demand for oil has all but dried up as lockdowns across the world have kept people inside.
As a result, oil firms have resorted to renting tankers to store the surplus supply and that has forced the price of US oil into negative territory.



Dynamics of Stock Prices

The current stock price $S(0)$ known to all investors is simply a positive number, but it can be thought of as a constant random variable. The unknown future prices $S(t)$ for $t > 0$ are non-constant random variables. This means that for each $t > 0$ there are at least two scenarios $\omega, \tilde{\omega} \in \Omega$ such that

$$S(t, \omega) = S(t, \tilde{\omega}).$$

Dynamics of Stock Prices

We assume that time runs in a discrete manner, $t = n\tau$, where $n = 0, 1, 2, 3, ...$ and τ is a fixed time step, typically a year, a month, a week, a day, or even a minute or a second to describe some hectic trading.
Because we take one year as the unit measure of time, a month corresponds to $\tau = \frac{1}{12}$, a week corresponds to $\tau = \frac{1}{52}$, a day to $\tau = \frac{1}{365}$, and so on.
To simplify our notation we shall write

$$S(0), S(1), S(2), ..., S(n), ...$$

instead of

$$S(0), S(\tau), S(2\tau), ..., S(n\tau), ...,$$

identifying n with $n\tau$. This convention will in fact be adopted for many other time-dependent quantities.

Dynamics of Stock Prices

Example

Suppose that there are three possible market scenarios, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, the stock prices taking the following values over two time steps:

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	55	58	60
ω_2	55	58	52
ω_3	55	52	53

Return

Return

It proves convenient to describe the dynamics of stock prices $S(n)$ in terms of returns. We assume that the stock pays no dividends.

Definition

The *rate of return*, or briefly the return $K(n,m)$ over a time interval $[n,m]$ (in fact $[m\tau, n\tau]$), is defined to be the random variable

$$K(n,m) = \frac{S(m) - S(n)}{S(n)}.$$

The return over a single time step $[n-1,n]$ will be denoted by $K(n)$, that is

$$K(n) = K(n-1,n) = \frac{S(n) - S(n-1)}{S(n-1)},$$

which implies that

$$S(n) = S(n-1)(1 + K(n)).$$

Return

Example

Suppose that there are three possible market scenarios, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, the stock prices taking the following values over two time steps:

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	55	58	60
ω_2	55	58	52
ω_3	55	52	53

The returns are random variables taking the following values:

Scenario	K(1)	K(2)	K(0,2)
ω_1	5.45%	3.45%	9%
ω_2	5.45%	−10.34%	5.45%
ω_3	−5.45%	1.92%	3.64%

Return

If the stock pays a dividend of $div(n)$ at time n, then the definition of return has to be modified. Typically, when a dividend is paid, the stock price drops by that amount. Since the right to a dividend is decided prior to the payment day, the drop of stock price is already reflected in $S(n)$. As a result, an investor who buys stock at time $n-1$ paying $S(n-1)$ and wishes to sell the stock at time n will receive $S(n) + div(n)$ and the return must reflect this:

$$K(n) = \frac{S(n) - S(n-1) + div(n)}{S(n-1)}.$$

Return

The precise relationship between consecutive one-step returns and the return over the aggregate period is

$$1 + K(n, m) = (1 + K(n+1))(1 + K(n+2))(1 + K(m)).$$

Expected Return

Suppose that the probability distribution of the return K over a certain time period is known. Then we can compute the mathematical expectation $E(K)$, called the *expected return.*

Proposition

If the one-step returns $K(n+1), \dots, K(m)$ are independent, then

$$1 + E(K(n,m)) = (1 + E(K(n+1)))(1 + E(K(n+2)))(1 + E(K(m))).$$

Binomial Tree Model

The model is defined by the following conditions.

- **Condition 1**
  The one-step returns $K(n)$ on stock are identically distributed independent random variables such that

$$K(n) = \begin{cases} u, & \text{with probability } p; \\ d, & \text{with probability } 1-p, \end{cases}$$

  at each time step n, where $1 < d < u$ and $0 < p < 1$.
- **Condition 2.**
  The one-step return r on a risk-free investment is the same at each time step and

$$d < r < u.$$

  The last condition describes the movements of stock prices in relation to risk-free assets such as bonds or cash held in a bank account.

Binomial Tree Model

The values of $S(n)$ along with the corresponding probabilities can be found for any n . In an n-step tree of stock prices each scenario (or path through the tree) with exactly i upward and $n-i$ downward price movements produces the same stock price

$$S(0)(1+u)^i(1+d)^{n-i} \text{ at time } n.$$

There are C_n^i such scenarios, the probability of each equal to $p^i(1-p)^{ni}$. As a result,

$$S(n) = S(0)(1+u)^i(1+d)^{n-i}$$

with probability $C_n^i p^i(1-p)^{n-i}$, for $i = 0, 1, ..., n$. The stock price $S(n)$ at time n is a discrete random variable with $n+1$ different values.

Binomial Tree Model

The number i of upward price movements is a random variable with a binomial distribution. The same is true for the number $n-i$ of downward movements. We therefore say that the price process follows a binomial tree. In an n-step binomial three the set Ω of all scenarios, that is, n-step paths moving up or down at each step has 2^n elements.

Risk-Neutral Probability

Risk-Neutral Probability

While the future value of stock can never be known with certainty, it is possible to work out expected stock prices within the binomial tree model. It is then natural to compare these expected prices and risk-free investments. This simple idea will lead us towards powerful and surprising applications in the theory of derivative securities.
To begin with, we shall work out the dynamics of expected stock prices $E(S(n))$. For $n = 1$

$$E(S(1)) = pS(0)(1 + u) + (1p)S(0)(1 + d) = S(0)(1 + E(K(1))),$$

where $E(K(1)) = pu + (1p)d$ is the expected one-step return. This extends to any n as follows.

Proposition

The expected stock prices for $n = 0, 1, 2, \ldots$ are given by

$$E(S(n)) = S(0)(1 + E(K(1)))^n.$$

Risk-Neutral Probability

If the amount $S(0)$ were to be invested risk-free at time 0, it would grow to $S(0)(1+r)^n$ after n steps. Clearly, to compare $E(S(n))$ and $S(0)(1+r)^n$ we only need to compare $E(K(1))$ and r.

An investment in stock always involves an element of risk, simply because the price $S(n)$ is unknown in advance.

A typical risk-averse investor will require that $E(K(1)) > r$, arguing that he or she should be rewarded with a higher expected return as a compensation for risk. The reverse situation when $E(K(1)) < r$ may nevertheless be attractive to some investors if the risky return is high with small non-zero probability and low with large probability.

The border case of a market in which $E(K(1)) = r$ is referred to as risk-neutral.

Risk-Neutral Probability

It proves convenient to introduce a special symbol p^* for the probability as well as E_* for the corresponding expectation satisfying the condition

$$E_*(K(1)) = p^* u + (1 - p^*) d = r$$

for risk-neutrality, which implies that

$$p^* = \frac{r - d}{u - d}.$$

We shall call p^* the risk-neutral probability and E_* the risk-neutral expectation. It is important to understand that p^* is an abstract mathematical object, which may or may not be equal to the actual market probability p. Only in a risk-neutral market do we have $p = p^*$. Even though the risk-neutral probability p^* may have no relation to the actual probability p, it turns out that for the purpose of valuation of derivative securities the relevant probability is p^*, rather than p.

Martingale Property

Martingale Property

The expectation of $S(n)$ with respect to the risk-neutral probability p^* is

$$E_*(S(n)) = S(0)(1+r)^n,$$

since $r = E(K(1))$.

Proposition

Given that the stock price $S(n)$ has become known at time n, the risk-neutral conditional expectation of $S(n+1)$ will be

$$E_*(S(n+1)|S(n)) = S(n)(1+r).$$

Thank you!