

Forward and Future contract. Option.

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Oslo 2022.9.12



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Introduction

Introduction

- The valuation of financial derivatives will be based on the principle of no arbitrage.

Arbitrage means making of a guaranteed risk free profit with a trade or a series of trades in the market. An **arbitrage free market** is a market which has no opportunities for risk free profit. An **arbitrage free price** for a security is a price that ensure that no arbitrage opportunity can be designed with that security.

- The principle of no arbitrage states that the markets must be arbitrage free.
- Some financial jargon will be used in what follows.

Financial jargon

- One says that has/takes a **long position** on an asset if one owns/is going to own a positive amount of that asset.
- One says that has/takes a **short position** on an asset if one has/is going to have a negative amount of that asset.
- Being short on money means borrowing. You can take a short position on many financial assets by **short selling**.

Example (Short selling)

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To implement some trading strategy you need to sell some amount of shares (to get money and invest in other assets).

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Example (Short selling)

To implement some trading strategy you need to sell some amount of shares (to get money and invest in other assets).

The problem is that you do not have any shares right now.

You can borrow the shares from another investor for a time period (paying interest) and sell the borrowed shares in the market.

At the end of the borrowing period you must buy again the shares in the market and give them back to the lender.

Forward Contracts

Forward contracts

A **forward contract** is an agreement to buy or sell an asset on a fixed date in the future, called the **delivery time**, for a price specified in advance, called the **forward price**.

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- Both parties are obliged to fulfill the terms of the contract.

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- The party selling the asset is said to be taking a **short forward position** and the party buying the asset is said to be taking a **long forward position**.
- Both parties are obliged to fulfill the terms of the contract.
- The main reason to enter into a forward contract agreement is to become independent of the unknown future price of a risky asset.
- Assume that the contract is entered at time $t = 0$, the delivery time is $t = T$ and denote by $F(0, T)$ the forward price.
- The time t price of the underlying asset is denoted $S(t)$.
- Due to the symmetry of the contract, no payment is made by either party at the beginning of the contract, $t = 0$.

Forward contracts

- At delivery, the party with the long position makes a profit if

$$F(0, T) < S(T),$$

while the party with the short position will make a loss (exactly of the same magnitude as the profit of the other party).

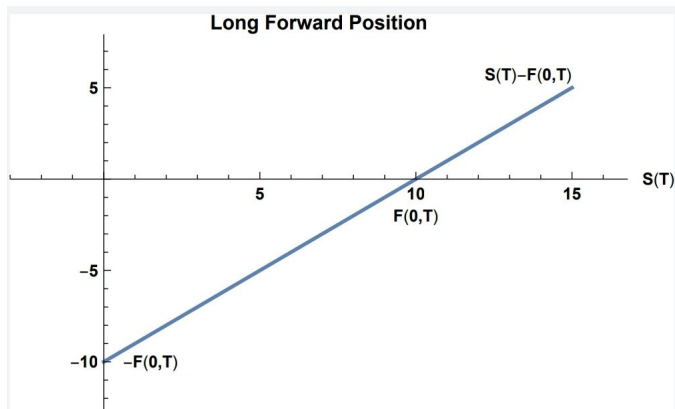
- If

$$F(0, T) > S(T)$$

the situation is reversed.

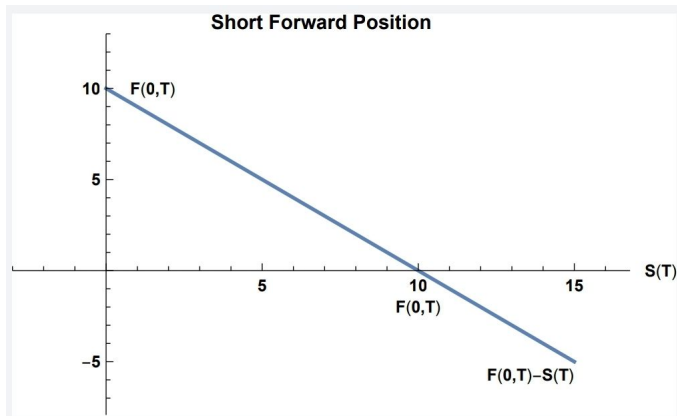
Forward contracts

The payoff at delivery for a long forward position is:



Forward contracts

- The payoff at delivery for a short forward position is:



Forward price

- If the contract is initiated at $t < T$, we will write $F(t, T)$ for the forward price and the payoffs will be $S(T) - F(t, T)$ (long position) and $F(t, T) - S(T)$ (short position).
- As no payment is made at the beginning of the forward contract, the main problem is to find the forward price $F(0, T)$ such that both parties are willing to enter into such agreement.
- One possible approach would be to compute the present value, which we know that is zero, by discounting the expected payoff of the contract.
- That is,

$$0 = V(0) = e^{-rT} \mathbb{E}[S(T) - F(0, T)],$$

which yields $F(0, T) = \mathbb{E}[S(T)]$.

- Note that $F(0, T)$ would depend on the distribution of $S(T)$, hence, we would only have translated the problem to agree on which distribution use.

Forward price

- The solution comes from the fact that we can also invest in the money market and there exists only one value for $F(0, T)$ such avoid arbitrages.
- The price obtained does not depend on the distribution of $S(T)$.

Buy and hold strategy

- Borrow $S(0)$ NOK, to buy the asset at time zero
- Hold it until time T .
- At time T , the amount $S(0)e^{rT}$ to be paid to settle the loan is a natural candidate for $F(0, T)$.

Forward price

The forward price $F(0, T)$ is given by

$$F(0, T) = S(0) e^{rT}, \quad (1)$$

where r is the constant risk free interest rate under continuous compounding. If the contract is initiated at time $t \leq T$, then

$$F(t, T) = S(t) e^{r(T-t)}. \quad (2)$$

Forward price

- The formula in the previous theorem applies as long as the underlying asset does not generate an income (dividends) or a cost (storage and insurance costs for commodities).
- In this lecture we will, many times, be implicitly assuming that the underlying is a stock which does not pay dividends.

In the case considered here we always have

$$F(t, T) = S(t) e^{r(T-t)} > S(t).$$

Moreover, the difference $F(t, T) - S(t)$, called the basis, converges to 0 as t converges to T .

Value of a forward contract

- Every forward contract has value zero when initiated.
- As time goes by, the price of the underlying asset may change and, along with it, the value of the forward contract.

The time t value of a long forward position with forward price $F(0, T)$ is given by

$$V(t) = (F(t, T) - F(0, T)) e^{-r(T-t)}, \quad 0 \leq t \leq T, \quad (3)$$

where $F(t, T)$ is the forward price of a contract starting at t and with delivery date T .

Value of a forward contract with forward price K

- Consider a forward contract with forward price K instead of $F(0, T)$.
- The value of this contract at time t will be given by equation (3) with $F(0, T)$ replaced by K ,

$$V_K(t) = (F(t, T) - K) e^{-r(T-t)}. \quad (4)$$

- Such contract may have non-zero initial value

$$V_K(0) = (F(0, T) - K) e^{-rT} = S(0) - Ke^{-rT}. \quad (5)$$

Futures

Futures

- One of the two parties in a forward contract will lose money.
- There is a risk of default (not being able to fulfill the contract) by the party losing money.
- Futures contracts are designed to eliminate such risk.

A **futures contract** is an exchange-traded standardized agreement between two parties to buy or sell an asset at an specified future time and at a price agreed today. The contract is marked to market daily and guaranteed by a clearing house.

- Assume that time is discrete with steps of length τ , typically one day.
- The market dictates the so called **futures prices** $f(n\tau, T)$ for each time step $\{n\tau\}_{n \geq 0}$ such that $n\tau \leq T$.
- These prices are random (and not known at time 0) except for $f(0, T)$.

Futures

- As in the case of forward contracts, it costs nothing to take a futures position.
- However, a futures contract involves a random cash flow, known as **marking to market**.
- Namely, at each time step $n\tau \leq T, n \geq 1$, the holder of a long futures position will receive the amount $f(n\tau, T) - f((n-1)\tau, T)$ if positive, or he/she will have to pay it if negative.
- The opposite payments apply for a short futures position.

The following conditions are imposed:

- 1 The futures price at delivery is $f(T, T) = S(T)$.
- 2 At each $n\tau \leq T, n \geq 1$, the value of a futures position is reset to zero, after marking to market.

Futures

- Each investor entering into a futures contract has to set up a deposit, called the **initial margin**, which is kept by the clearing house as collateral.
- In the case of a long futures position the amount is added/subtracted to the deposit, depending if it is positive/negative, at each time step. The opposite amount is added/subtracted for a short position.
- Any excess above the initial margin can be withdrawn.
- However, if the deposit drops below a certain level, called the **maintenance margin**, the clearing house will issue a **margin call**, requesting the investor to make a payment and restore the deposit to the level of the initial margin.
- If the investor fails to respond to a margin call the futures position is closed by the clearing house.

Futures

- Futures markets are very liquid (high number of transactions) due to standardization and the presence of a clearing house.
- There is no risk of default, in contrast to forward contracts negotiated directly between two parties.
- A negative aspect of the highly standardized contracts is that you may not find a contract that actually covers the risk you want to hedge.
- For example, you want protection against adverse movements of the share prices of a company, but in the market there are no futures on the share price of that company.
- In this case, you may use futures on a stock index containing the company, but the hedge is less perfect and more risky.

Futures

If the interest rate r is constant, then $f(0, T) = F(0, T)$.

- In an economy with constant interest rate r we obtain a simple structure of futures prices

$$f(t, T) = S(t) e^{r(T-t)}. \quad (6)$$

- Note that the futures are random but only due to $S(t)$.

Options

Options

A **European call/put option** is a contract giving the holder the right (but not the obligation) to buy/sell an asset for a price K fixed in advance, known as the **exercise price** or **strike price**, at a specified future date T , called the **exercise time** or **expiry date**. An **American call/put option** is a contract giving the holder the right (but not the obligation) to buy/sell an asset for a strike price K , also fixed in advance, at any time between now and the expiry date T .

- Some underlying assets may be impossible to buy or sell (e.g., stock indices).
- The option then is cleared in cash.

Options

- An option is determined by its **payoff**, which for a European call is

$$(S(T) - K)^+ = \max(0, S(T) - K) = \begin{cases} S(T) - K & \text{if } S(T) > K \\ 0 & \text{if } S(T) \leq K \end{cases} ,$$

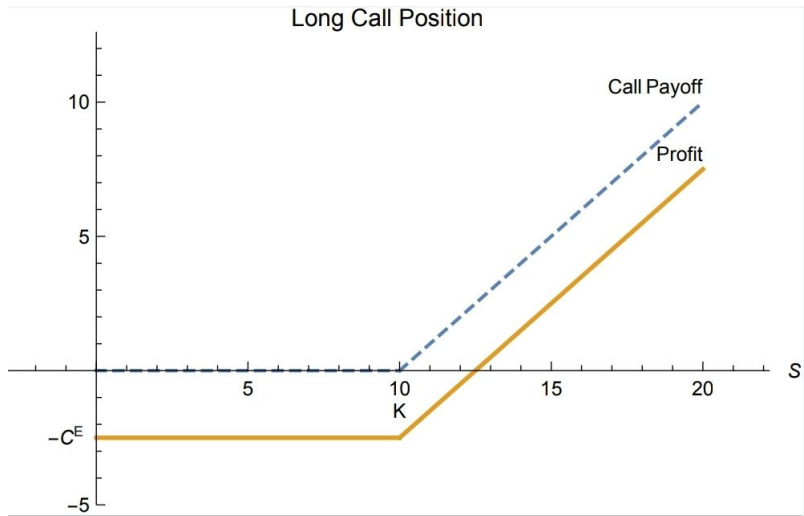
and for a put option is $(K - S(T))^+$.

- Since the payoffs are non negative, a premium must be paid to buy an option, otherwise there is an arbitrage opportunity.
- The **premium** is the market price of the option at time 0.

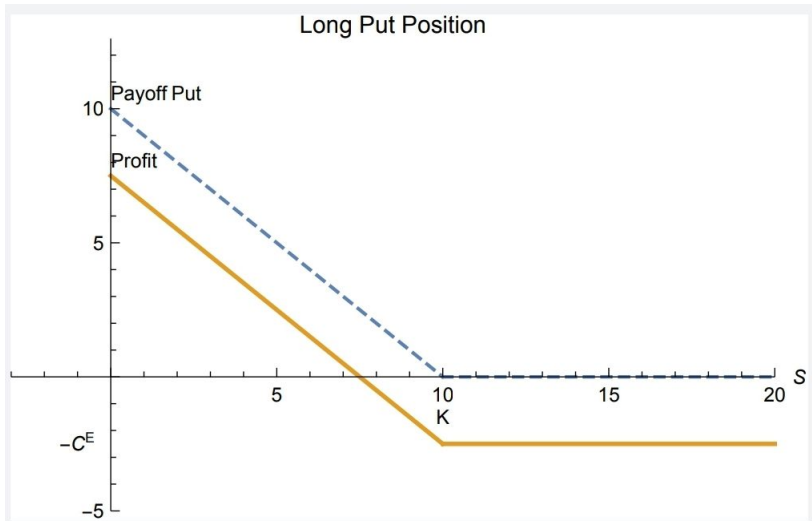
Options

- The prices of European calls and puts will be denoted by C^E and P^E .
- We will use the notation C^A and P^A for the American ones.
- The same constant interest rate r will apply for borrowing and lending money and we will use continuous compounding.
- The **gain/profit of an option** buyer (seller, also known as writer) is the payoff minus (plus) the premium C^E or P^E paid (received) for the option.

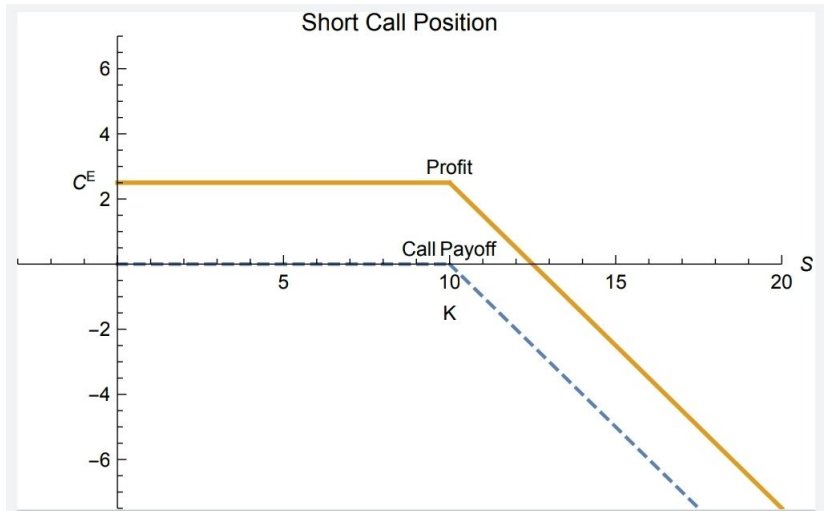
Options



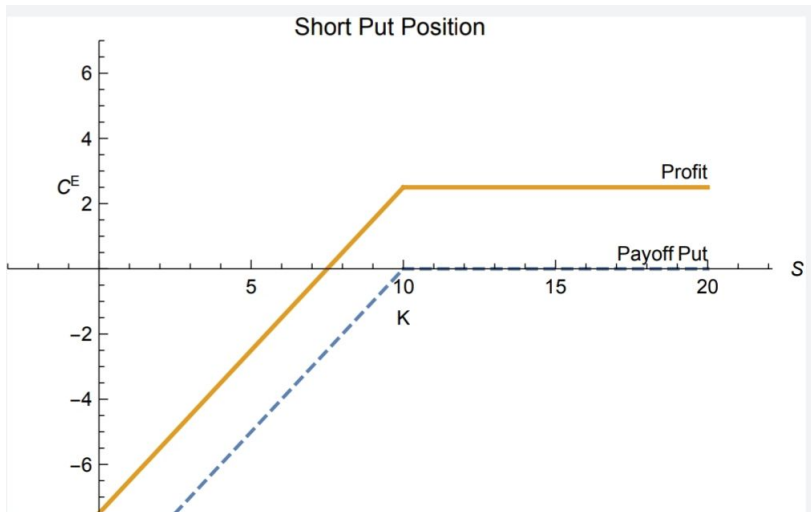
Options



Options



Options



Put-call parity

- In what follows we will find bounds and some general properties for option prices.
- We will use the principle of no arbitrage alone, without any assumption on the evolution of the underlying asset prices.
- Take a long position with a European call and a short position with a European put, both with the same strike K and expiry time T .
- We obtain a portfolio having the same payoff as a long forward position with forward price K and delivery time T , that is,

$$(S(T) - K)^+ - (K - S(T))^+ = S(T) - K.$$

- As a result, the present value of such portfolio of options should be that of a forward contract with forward price K , which is

$$S(0) - Ke^{-rT}.$$

Put-call parity

[European put-call parity] For a stock paying no dividends, the prices of European call and put options, both with the same strike price K and exercise time T , satisfy

$$C^E - P^E = S(0) - Ke^{-rT}. \quad (7)$$

Put-call parity

[American put-call parity estimates] For a stock paying no dividends, the price of American call and put options, both with the same strike price K and expiry time T , satisfy

$$C^A - P^A \leq S(0) - Ke^{-rT}, \quad (8)$$

$$C^A - P^A \geq S(0) - K. \quad (9)$$

Proof.

Smartboard □

Bounds on option prices

- We will assume that all the options have the same strike K and expiry time T .
- We start by noting the following obvious inequalities

$$\left. \begin{aligned} 0 \leq C^E \leq C^A, \\ 0 \leq P^E \leq P^A. \end{aligned} \right\} \quad (10)$$

- The option prices must be non-negative because they have non-negative payoff.
- American options should be more expensive because they give at least the same rights as their European counterparts.

Bounds on option prices

On a stock paying no dividends one has that

$$\left(S(0) - Ke^{-rT}\right)^+ = \max\left(0, S(0) - Ke^{-rT}\right) \leq C^E < S(0), \quad (11)$$

$$\left(Ke^{-rT} - S(0)\right)^+ = \max\left(0, Ke^{-rT} - S(0)\right) \leq P^E < Ke^{-rT}. \quad (12)$$

Proof.

Smartboard □

On a stock paying no dividends one has that

$$C^E = C^A. \quad (13)$$

Proof.

Smartboard. □

Bounds on option prices

- As $C^A \geq C^E$ and $C^E \geq S(0) - Ke^{-rT}$, it follows that

$$C^A > S(0) - K,$$

if $r > 0$.

- Because the price of the American option is greater than its payoff, the option will sooner be sold than exercised at time 0.
- Similar inequalities hold for $t < T$ and one can repeat the arguments to conclude that the American option will never be exercised prior to the expiry time.
- This also shows that the American option is equivalent to the European option.

Bounds on option prices

On a stock paying no dividends one has that

$$\left(S(0) - Ke^{-rT}\right)^+ = \max\left(0, S(0) - Ke^{-rT}\right) \leq C^A < S(0), \quad (14)$$

$$(K - S(0))^+ = \max\left(0, Ke^{-rT} - S(0)\right) \leq P^A < K. \quad (15)$$

Proof.

Exercise. □

Variables determining option prices

- Here we will study how the option prices depend on variables such the strike K , the current price of the underlying $S(0)$, and the expiry time T .
- We shall analyse option prices as functions of one of the variables, keeping the remaining variables constant.

European options: dependence on the strike price

If $K_1 < K_2$, then

1 Monotonicity: $C^E(K_1) > C^E(K_2)$ and $P^E(K_1) < P^E(K_2)$.

2 Lipschitz continuity:

$$C^E(K_1) - C^E(K_2) < e^{-rT} (K_2 - K_1),$$

$$P^E(K_2) - P^E(K_1) < e^{-rT} (K_2 - K_1).$$

3 Convexity: For $\alpha \in (0, 1)$ we have

$$C^E(\alpha K_1 + (1 - \alpha) K_2) \leq \alpha C^E(K_1) + (1 - \alpha) C^E(K_2),$$

$$P^E(\alpha K_1 + (1 - \alpha) K_2) \leq \alpha P^E(K_1) + (1 - \alpha) P^E(K_2).$$

Proof.

Smartboard. □

European options: dependence on the underlying asset price

- The current price $S(0)$ of the underlying asset is given by the market and cannot be changed.
- But we can consider an option on a portfolio of x shares, worth $S = xS(0)$.
- The payoff of a European call with strike K on such portfolio, to be exercised at time T , will be $(xS(T) - K)^+$.
- We shall study the dependence of option prices on S .
- We will denote the call and put prices by $C^E(S)$ and $P^E(S)$.

European options: dependence on the underlying asset price

If $S_1 < S_2$, then

- 1 **Monotonicity:** $C^E(S_1) < C^E(S_2)$ and $P^E(S_1) > P^E(S_2)$.
- 2 **Lipschitz continuity:**

$$C^E(S_2) - C^E(S_1) < S_2 - S_1,$$

$$P^E(S_1) - P^E(S_2) < S_2 - S_1.$$

- 3 **Convexity:** For $\alpha \in (0, 1)$ we have

$$C^E(\alpha S_1 + (1 - \alpha) S_2) \leq \alpha C^E(S_1) + (1 - \alpha) C^E(S_2),$$

$$P^E(\alpha S_1 + (1 - \alpha) S_2) \leq \alpha P^E(S_1) + (1 - \alpha) P^E(S_2).$$

Proof.

Smartboard.



American options: dependence on the strike price

If $K_1 < K_2$, then

- 1 Monotonicity: $C^A(K_1) > C^A(K_2)$ and $P^A(K_1) < P^A(K_2)$.
- 2 Lipschitz continuity:

$$\begin{aligned}C^A(K_1) - C^A(K_2) &< (K_2 - K_1), \\P^A(K_2) - P^A(K_1) &< (K_2 - K_1).\end{aligned}\tag{16}$$

- 3 Convexity: For $\alpha \in (0, 1)$ we have

$$\begin{aligned}C^A(\alpha K_1 + (1 - \alpha) K_2) &\leq \alpha C^A(K_1) + (1 - \alpha) C^A(K_2), \\P^A(\alpha K_1 + (1 - \alpha) K_2) &\leq \alpha P^A(K_1) + (1 - \alpha) P^A(K_2).\end{aligned}$$

Proof.

Smartboard. □

American options: dependence on the underlying asset price

- As in the European case, we consider options on a portfolio of x shares worth $S = xS(0)$.

If $S_1 < S_2$ then

1 **Monotonicity:** $C^A(S_1) < C^A(S_2)$ and $P^A(S_1) > P^A(S_2)$.

2 **Lipschitz continuity:**

$$C^A(S_2) - C^A(S_1) < S_2 - S_1,$$

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$$C^E(\alpha S_1 + (1 - \alpha) S_2) \leq \alpha C^E(S_1) + (1 - \alpha) C^E(S_2),$$

$$P^E(\alpha S_1 + (1 - \alpha) S_2) \leq \alpha P^E(S_1) + (1 - \alpha) P^E(S_2).$$

American options: dependence on expiry time

If $T_1 < T_2$, then $C^A(T_1) \leq C^A(T_2)$ and $P^A(T_1) \leq P^A(T_2)$.

Proof: (only for calls, the proof for puts being analogous) .

Suppose that $C^A(T_1) > C^A(T_2)$, then

- Sell the option with shorter time to expiry and buy the one with longer time to expiry, investing the balance risk free.
 - If the option sold is exercised at time $t \leq T_1$, we can exercise the other option to cover our liability.
 - The risk-less profit will be $(C^A(T_1) - C^A(T_2)) e^{rt} > 0$.



The previous arguments do not work for European options because early exercise is not possible.

Thank you!