Option. Main properties

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Oslo 2022.9.19

Contents

Put-call parity

Put-call parity

[American put-call parity estimates] For a stock paying no dividends, the price of American call and put options, both with the same strike price K and expiry time T, satisfy

$$C^A - P^A \leq S(0) - Ke^{-rT}, \quad (1)$$

$$C^A - P^A \geq S(0) - K. \quad (2)$$

Proof.

□

Bounds on Option Prices

Bounds on option prices

- We will assume that all the options have the same strike K and expiry time T.
- We start by noting the following obvious inequalities

$$\left.\begin{array}{l} 0 \leq C^E \leq C^A, \\ 0 \leq P^E \leq P^A. \end{array}\right\} \tag{3}$$

- The option prices must be non-negative because they have non-negative payoff.
- American options should be more expensive because they give at least the same rights as their European counterparts.

Bounds on option prices

On a stock paying no dividends one has that

$$\left(S(0) - Ke^{-rT}\right)^{+} = \max\left(0, S(0) - Ke^{-rT}\right) \le C^{E} < S(0), \tag{4}$$

$$\left(Ke^{-rT} - S(0)\right)^{+} = \max\left(0, Ke^{-rT} - S(0)\right) \le P^{E} < Ke^{-rT}. \tag{5}$$

Proof.

Smartboard □

On a stock paying no dividends one has that

$$C^{E} = C^{A}. \tag{6}$$

Proof.

Smartboard. □

Bounds on option prices

- As $C^A \geq C^E$ and $C^E \geq S(0) - Ke^{-rT}$, it follows that

$$C^A > S(0) - K,$$

  if $r > 0$.
- Because the price of the American option is greater than its payoff, the option will sooner be sold than exercised at time 0.
- Similar inequalities hold for $t < T$ and one can repeat the arguments to conclude that the American option will never be exercised prior to the expiry time.
- This also shows that the American option is equivalent to the European option.

Bounds on option prices

On a stock paying no dividends one has that

$$\left(S(0) - Ke^{-rT}\right)^{+} = \max\left(0, S(0) - Ke^{-rT}\right) \le C^{A} < S(0), \tag{7}$$

$$(K - S(0))^{+} = \max\left(0, Ke^{-rT} - S(0)\right) \le P^{A} < K. \tag{8}$$

Proof.

Exercise. □

Variables Determining Option Prices

Variables determining option prices

- Here we will study how the option prices depend on variables such the strike K, the current price of the underlying $S\left(0\right)$, and the expiry time T.
- We shall analyse option prices as functions of one of the variables, keeping the remaining variables constant.

European options: dependence on the strike price

If $K_1 < K_2$, then

1. Monotonicity: $C^E(K_1) > C^E(K_2)$ and $P^E(K_1) < P^E(K_2)$.
2. Lipschitz continuity:

$$C^E(K_1) - C^E(K_2) < e^{-rT}(K_2 - K_1),$$
$$P^E(K_2) - P^E(K_1) < e^{-rT}(K_2 - K_1).$$

3. Convexity: For $\alpha \in (0,1)$ we have

$$C^E(\alpha K_1 + (1-\alpha) K_2) \leq \alpha C^E(K_1) + (1-\alpha) C^E(K_2),$$
$$P^E(\alpha K_1 + (1-\alpha) K_2) \leq \alpha P^E(K_1) + (1-\alpha) P^E(K_2).$$

Proof.

□

European options: dependence on the underlying asset price

- The current price $S(0)$ of the underlying asset is given by the market and cannot be changed.
- But we can consider an option on a portfolio of x shares, worth $S = xS(0)$.
- The payoff of a European call with strike K on such portfolio, to be exercised at time T, will be $(xS(T) - K)^+$.
- We shall study the dependence of option prices on S.
- We will denote the call and put prices by $C^E(S)$ and $P^E(S)$.

European options: dependence on the underlying asset price

If $S_1 < S_2$, then

1. Monotonicity: $C^E(S_1) < C^E(S_2)$ and $P^E(S_1) > P^E(S_2)$.
2. Lipschitz continuity:

$$C^E(S_2) - C^E(S_1) < S_2 - S_1,$$
$$P^E(S_1) - P^E(S_2) < S_2 - S_1.$$

3. Convexity: For $\alpha \in (0,1)$ we have

$$C^E(\alpha S_1 + (1-\alpha) S_2) \leq \alpha C^E(S_1) + (1-\alpha) C^E(S_2),$$
$$P^E(\alpha S_1 + (1-\alpha) S_2) \leq \alpha P^E(S_1) + (1-\alpha) P^E(S_2).$$

Proof.

□

American options: dependence on the strike price

If $K_1 < K_2$, then

1. Monotonicity: $C^A(K_1) > C^A(K_2)$ and $P^A(K_1) < P^A(K_2)$.
2. Lipschitz continuity:

$$\begin{aligned} C^A(K_1) - C^A(K_2) &< (K_2 - K_1), \\ P^A(K_2) - P^A(K_1) &< (K_2 - K_1). \end{aligned} \tag{9}$$

3. Convexity: For $\alpha \in (0,1)$ we have

$$\begin{aligned} C^A(\alpha K_1 + (1-\alpha)K_2) &\leq \alpha C^A(K_1) + (1-\alpha)C^A(K_2), \\ P^A(\alpha K_1 + (1-\alpha)K_2) &\leq \alpha P^A(K_1) + (1-\alpha)P^A(K_2). \end{aligned}$$

Proof.

□

American options: dependence on the underlying asset price

- As in the European case, we consider options on a portfolio of x shares worth $S = xS(0)$.

If $S_1 < S_2$ then

1. Monotonicity: $C^A(S_1) < C^A(S_2)$ and $P^A(S_1) > P^A(S_2)$.
2. Lipschitz continuity:

$$C^A(S_2) - C^A(S_1) < S_2 - S_1,$$
$$P^A(S_1) - P^A(S_2) < S_2 - S_1.$$

3. Convexity: For $\alpha \in (0,1)$ we have

$$C^E(\alpha S_1 + (1-\alpha) S_2) \leq \alpha C^E(S_1) + (1-\alpha) C^E(S_2),$$
$$P^E(\alpha S_1 + (1-\alpha) S_2) \leq \alpha P^E(S_1) + (1-\alpha) P^E(S_2).$$

American options: dependence on expiry time

If $T_1 < T_2$, then $C^A(T_1) \leq C^A(T_2)$ and $P^A(T_1) \leq P^A(T_2)$.

Proof: (only for calls, the proof for puts being analogous) .

Suppose that $C^A(T_1) > C^A(T_2)$, then

- Sell the option with shorter time to expiry and buy the one with longer time to expiry, investing the balance risk free.
  - If the option sold is exercised at time $t \leq T_1$, we can exercise the other option to cover our liability.
  - The risk-less profit will be $\left(C^A(T_1) - C^A(T_2)\right) e^{rt} > 0$.

□

The previous arguments do not work for European options because early exercise is not possible.

Time Value of Options

Time value of options

We say that at time $0 \leq t \leq T$ a call option with strike K is

- ***(deep) in the money*** if $S(t) \overset{(\gg)}{>} K$,
- ***at the money*** if $S(t) = K$,
- ***(deep) out of the money*** if $S(t) \overset{(\ll)}{<} K$.

The same terminology applies to put options but with the inequalities reversed. At time $0 \leq t \leq T$, the ***intrinsic value of a call (put) option*** with strike K is equal to $(S(t) - K)^+$ ($(K - S(t))^+$).

Time value of options

- The intrinsic value of out of the money or at the money options is zero.
- Options in the money have positive intrinsic value.
- The price of an American option prior to expiry must be greater than its intrinsic value.
- The price of a European option prior to expiry may be greater or smaller than its intrinsic value.

Time value of options

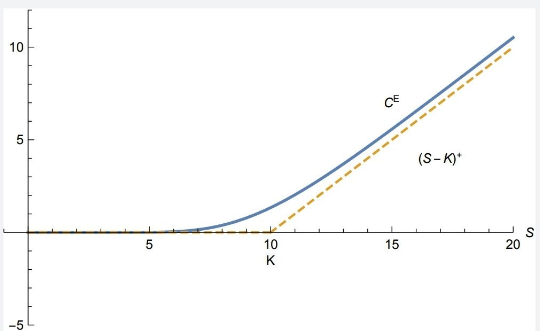
The ***time value of an option*** is the difference between the price of the option and its intrinsic value, that is,

$$
\begin{aligned}
&C^E(t) - (S(t) - K)^+, && \text{European call}\\
&P^E(t) - (K - S(t))^+, && \text{European put}\\
&C^A(t) - (S(t) - K)^+, && \text{American call}\\
&P^A(t) - (K - S(t))^+. && \text{American put.}
\end{aligned}
$$

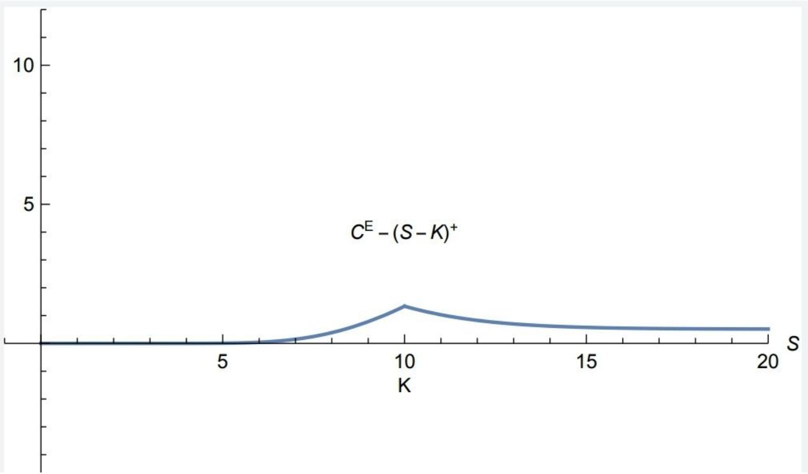
Here, the argument t in the option prices denotes the current time and NOT the expiry time as in Proposition 17.

Time value of options

- The time value of a European call as a function of $S(t)$ is always nonnegative.
- For in the money calls, the time value is bigger than $K - Ke^{-r(T-t)}$, due to the inequality $C^E(t) \geq S(t) - Ke^{-r(T-t)}$
- The same applies to an American call because their prices coincide.

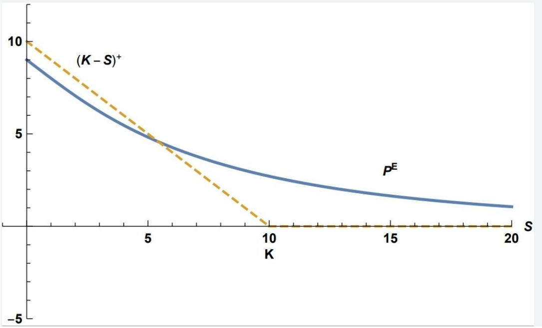


Time value of options

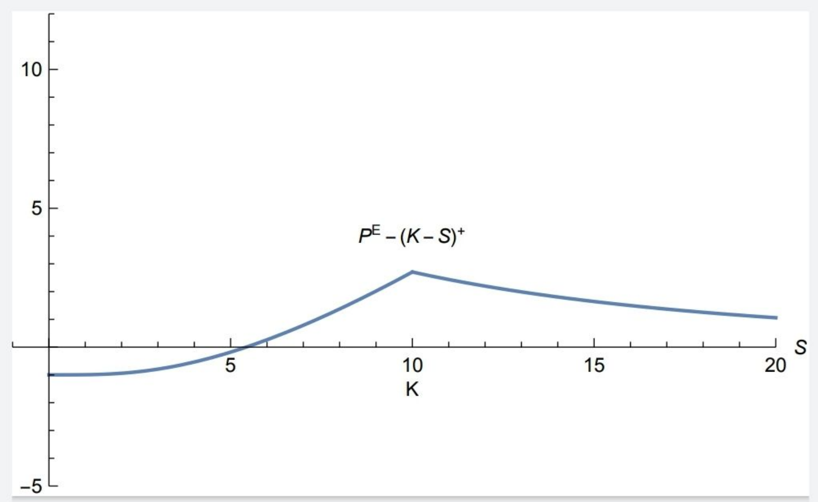


Time value of options

- The time value of a European put may be negative.
- This happens if the put option is deep in the money, because we can only exercise the option at time T and there is a considerable risk that in the meanwhile the stock price rises.

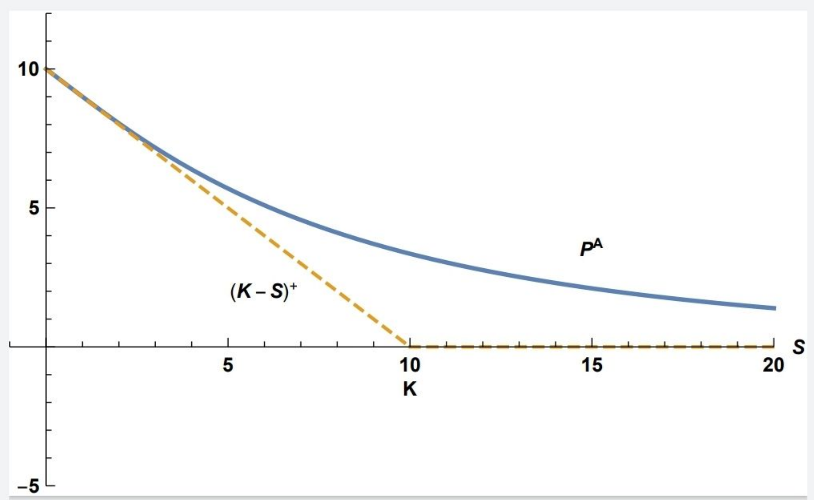


Time value of options

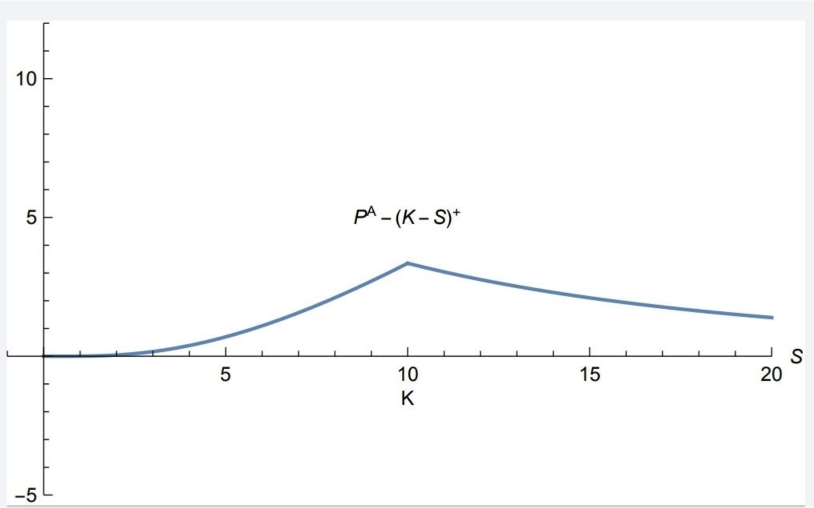


Time value of options

- The time value of an American put is always nonnegative.



Time value of options



Time value of options

For any European or American call or put with strike price K, the time value attains its maximum at $S = K$.

Proof. (only for European calls).

If $S \leq K$ the intrinsic value is zero. Since $C^E(S)$ is an increasing function of S, this means that the time value of the call is increasing for $S \leq K$. If $K \leq S_1 < S_2$, we have that $C^E(S_2) - C^E(S_1) \leq S_2 - S_1$ and, hence,

$$
\begin{aligned}
C^E(S_2) - S_2 &\leq C^E(S_1) - S_1 \\
&\Updownarrow \\
C^E(S_2) - (S_2 - K)^+ &= C^E(S_2) - S_2 + K \\
&\leq C^E(S_1) - S_1 + K = C^E(S_1) - (S_1 - K)^+ ,
\end{aligned}
$$

which yields that the time value of the call is a decreasing function of S if $S \geq K$. Therefore, the maximum is at $S = K$. □

Hedging and speculating with options

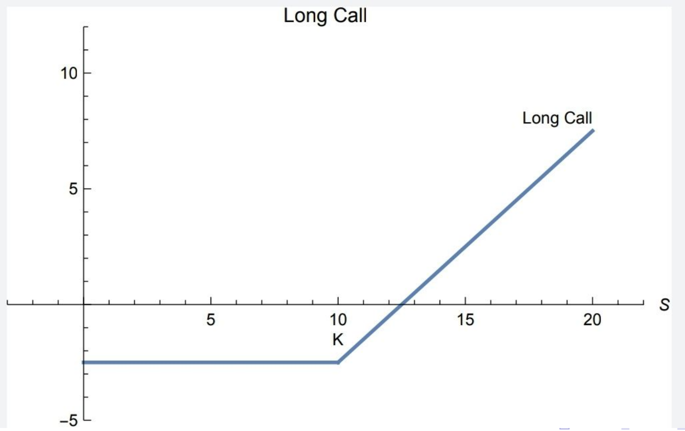
- Suppose that you are an investor with specific views on the future behavior of stock prices and you are willing to take/avoid risks.
- Using European options, we will show some of the most popular strategies, used to hedge/speculate on stock prices.
- These strategies can be classified into three categories:
  - ***Bullish***: These strategies are used when a rise in the price of the stock is expected.
  - ***Bearish***: These strategies are used when a fall in the prices of the stock is expected.
  - ***Neutral or non-directional***: These strategies are used when no clear direction in the price of the stock is expected. They bet on the volatility (standard deviation) of the stock price.

Hedging and speculating with options

- We will assume that all the options are on the same stock.
- We will also assume have the same strike K and expiry time T, unless stated otherwise.
- By building portfolios of calls, puts, underlying and bonds, you can replicate any piecewise linear terminal payoff function.
- Thanks to the put-call parity different combinations of calls, puts, stocks and risk free investment may produce the same terminal profits.
- Hence, there is no unique way of implementing the following strategies.
- We will use a solid line to plot the total profit and dashed lines to plot individual option profits.

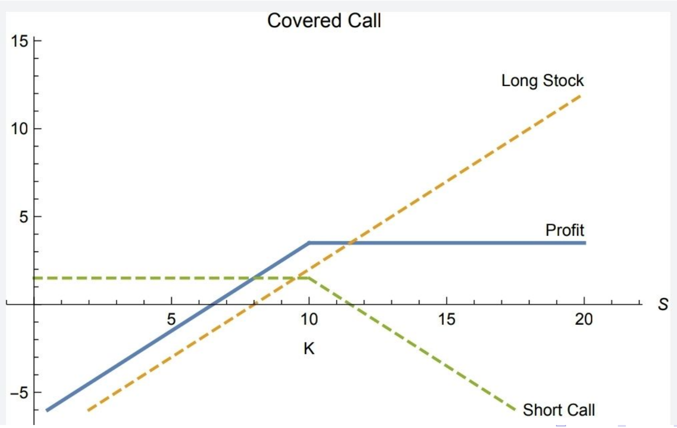
Bullish strategies: long call

- Buy a call option.
- You expect a high rise in the price of the stock.
- Loses are limited.



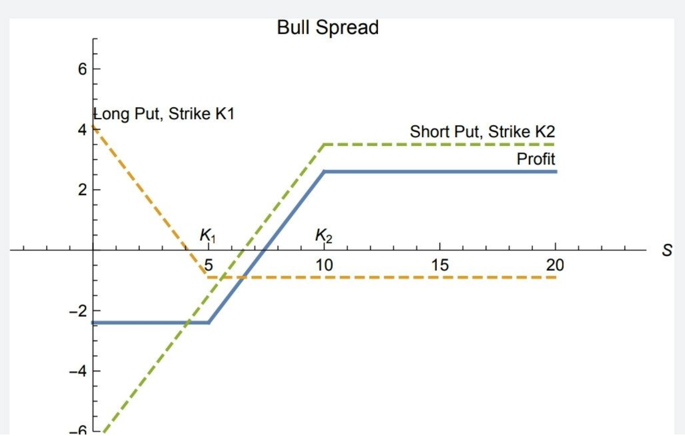
Bullish strategies: covered call

- Sell a call option and buy the stock.
- You expect a moderate rise in the price of the stock.
- Loses can be very high.



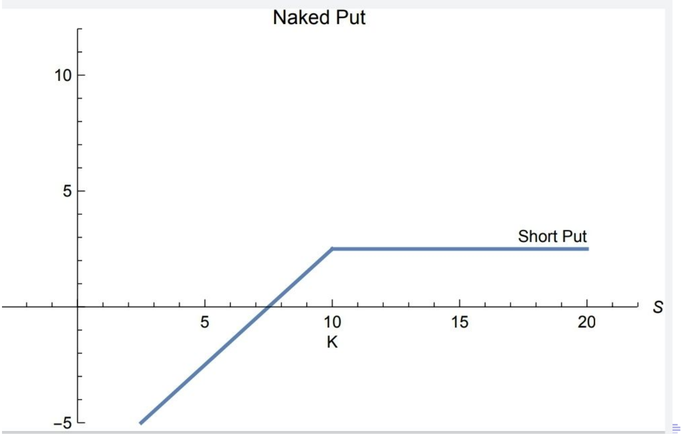
Bullish strategies: bull spread

- Buy a put option with strike K_1 and sell a put option with higher strike K_2.
- You expect a moderate rise in the price of the stock.
- Loses are limited.



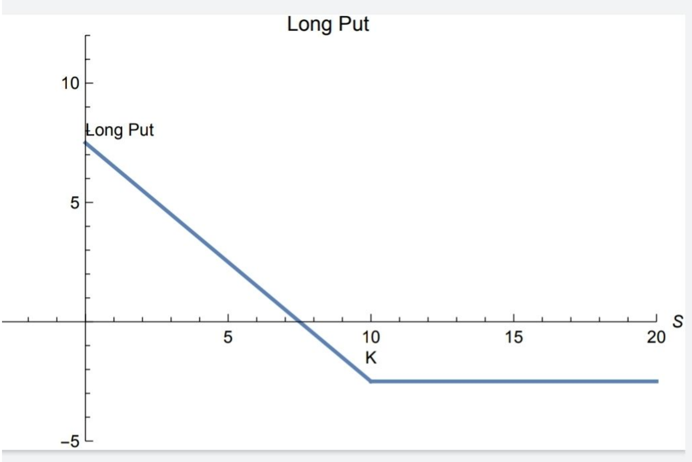
Bullish strategies: naked put

- You sell a put option with strike K.
- You expect that at expiry time S will be above K .
- Loses can be very high.



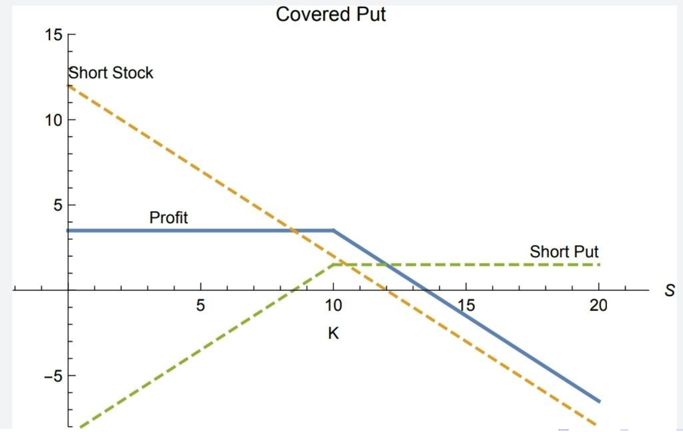
Bearish strategies: long put

- You buy a put option with strike K.
- You expect a big drop in the price of the stock.
- Loses are limited.



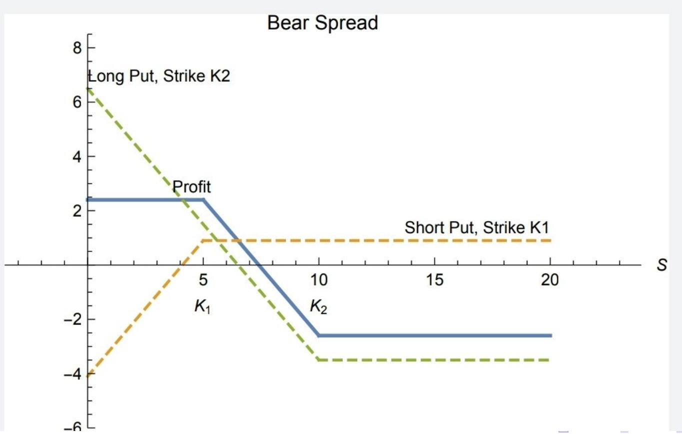
Bearish strategies: covered put

- You sell a put option and sell the stock.
- You expect a moderate drop in the price of the stock.
- Loses can be very high.



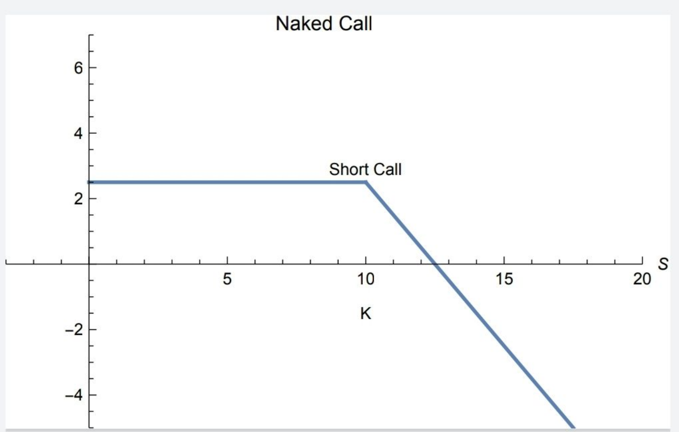
Bearish strategies: bear spread

- You buy a put option with strike K_2 and sell a put option with lower strike K_1.
- You expect a moderate drop in the price of the stock.
- Loses are limited.



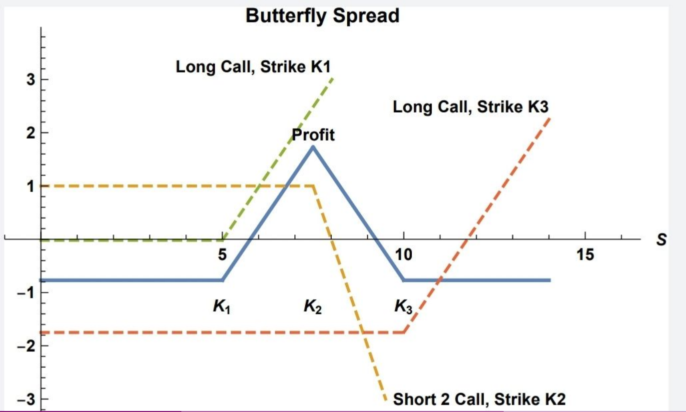
Bearish strategies: naked call

- You sell a call option with strike K.
- You expect that at expiry time S will be below K .
- Loses can be very high. (Unbounded, in theory)



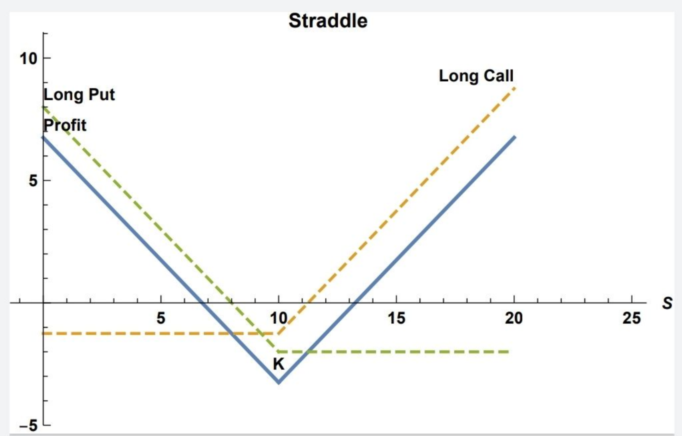
Neutral strategies: butterfly spreads

- Let $0 < K_1 < K_2 < K_3$. Buy a call option with strike K_1 and a call option with strike K_3 and sell two call options with strike K_2.
- You make a profit if the S stays close to K_2.
- Hence, you expect a low volatility for the stock price.
- Loses are limited.



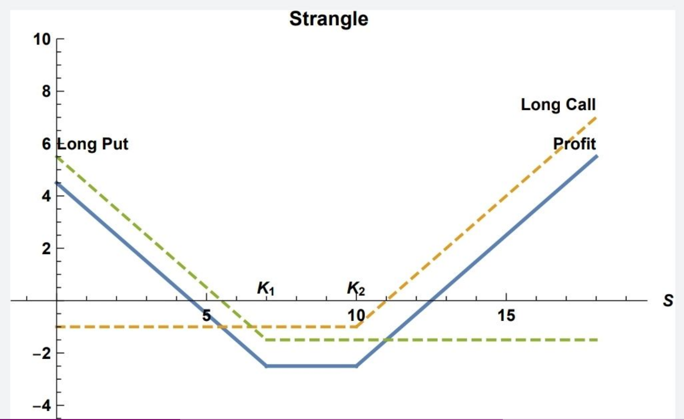
Neutral strategies: straddles

- You buy a call option and a put option with the same strike K.
- You make a profit if S ends far away from K.
- Hence, you expect a high volatility for the stock price.
- Loses are limited.



Neutral strategies: strangles

- Let $0 < K_1 < K_2$. Buy a call option with strike K_2 and a put option with strike K_1.
- You make a profit if S ends far out of the interval $[K_1, K_2]$.
- Hence, you expect a very high volatility for the stock price.
- Loses are limited. Strangles are cheaper than straddles.



Thank you!