Single Period Financial Markets. Model specifications.

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Introduction

Single period models are

- Unrealistic (prices change almost continuously in time)
- Mathematically simple (linear algebra + discrete probability)
- Useful (easily illustrate many economic principles observed in real markets)

Model specifications

A single period model of financial markets is specified by the following ingredients:

1. ***Initial date*** $(t = 0)$ and a ***terminal date*** $(t = 1)$.
2. A ***finite sample space*** $\Omega = \{\omega_1, ..., \omega_K\}$ with $K \in \mathbb{N}$.
   - Each ω represents a possible state of the economy/world. (mutually exclusive)
   - At $t = 0$ the investor does not know the state of the world.
   - Financial assets have a constant value at $t = 0$, but its value will depend on $\omega \in \Omega$ at time $t = 1$. (random variables)
3. A ***probability measure*** P (that is, a function $P : \Omega \to [0, 1]$ with $\sum_{i=1}^{K} P(\omega_i) = 1$), which we additionally assume to satisfy $P(\omega) > 0, \omega \in \Omega$.
4. A ***bank account process*** $B = \{B(t)\}_{t=0,1} = \{B(0), B(1)\}$, where with $B(0) = 1$ and $B(1)$ is a random variable with $B(1, \omega) > 0$.
   In fact, one usually finds that $B(1) \geq 1$.

Model specifications

Definition 1 (continuation)

Then, one has that

$$r = \left(B\left(1\right) - B\left(0\right)\right)/B\left(0\right) = B\left(1\right) - 1 \geq 0.$$

Moreover, a usual assumption is that $B\left(1\right)$ and r are constants.

5. A ***price process*** $S = \left\{S\left(t\right)\right\}_{t=0,1} = \left\{S\left(0\right), S\left(1\right)\right\}$ where

$$S\left(t\right) = \left(S_1\left(t\right), \cdots, S_N\left(t\right)\right)^T,$$

and $N \geq 1$ is the number of risky assets.
You may think of these assets as stocks.

- At $t = 0$: the investor knows the value of the stocks, i.e., $S\left(0\right)$ are constants.
- At $t = 1$: the prices $S\left(1\right)$ are random variables, whose actual realizations become known to the investor only at time $t = 1$.

Model specifications

Definition 1 (continuation)

S represents the price of the risky assets because, usually, for all $j = 1, ..., N$ there exists $\omega_1(j)$ and $\omega_2(j)$ in Ω such that

$$S_j(1, \omega_1(j)) < S_j(0) < S_j(1, \omega_2(j)).$$

Note that $S_j(0) = S_j(0, \omega), \omega \in \Omega$, because $S_j(0)$ is constant.

Model specifications

A ***trading strategy*** is a vector $H = (H_0, H_1, \cdots, H_N)^T$, where

- $H_0 :=$ Amount of money invested in the bank account.
- $H_n :=$ Number of units of security n held between $t = 0$ and $t = 1$, $n = 1, \ldots, N$.
- Note that $H_n, n = 0, \ldots, N$ can be negative: borrowing/short selling.
- Moreover, $H_n, n = 0, \ldots, N$ are constants because these are decision taken at $t = 0$.

Model specifications

The ***value process*** $V = \{V(t)\}_{t=0,1}$, is the total value of the portfolio, associated to a trading strategy H, at each t, which is given by

$$V(t) = H_0 B(t) + \sum_{n=1}^{N} H_n S_n(t), \qquad t = 0, 1. \tag{1}$$

- Note that $V(0)$ is constant and $V(1)$ is a random variable.

Model specifications

The ***gain process*** G is the random variable describing the total profit/loss generated by a trading strategy H between $t = 0$ and $t = 1$ and is given by

$$G = H_0 \left(B\left(1\right) - B\left(0\right) \right) + \sum_{n=1}^{N} H_n \left(S_n\left(1\right) - S_n\left(0\right) \right)$$
$$= H_0 r + \sum_{n=1}^{N} H_n \Delta S_n. \qquad (2)$$

- Note that
$$V\left(1\right) = V\left(0\right) + G. \qquad (3)$$
- Moreover, the change in V is due to the changes in S, no addition/withdraw of funds allowed.

Model specifications

A ***numeraire*** is a financial asset used to measure the value of all other assets in the market, i.e., the price of all financial assets are expressed in units of numeraire.

- We will use the bank account as numeraire.
- As a consequence, $B(t) = 1, t = 0, 1$, and the quantities S, V and G will have their discounted versions (***normalized market***).

The discounted price process $S^* = \{S^*(t)\}_{t=0,1}$ is given by

$$S_n^*(t) = \frac{S_n(t)}{B(t)}, \qquad n = 1, ..., N, t = 0, 1. \tag{4}$$

Model specifications

The ***discounted value process*** $V^* = \{V^*(t)\}_{t=0,1}$ is given by

$$V^*(t) = \frac{V(t)}{B(t)}, \qquad n = 1, ..., N, t = 0, 1. \tag{5}$$

The ***discounted gains process*** G^* is given by

$$G^* = H_0 (B^*(1) - B^*(0)) + \sum_{n=1}^{N} H_n (S_n^*(1) - S_n^*(0)) = \sum_{n=1}^{N} H_n \Delta S_n^*. \tag{6}$$

Moreover,

$$V^*(1) = V^*(0) + G^* \tag{7}$$

Model specifications

In a single period financial market model with $\#\Omega = K$ and N risky assets, the ***payoff matrix*** $S(1, \Omega)$ is defined to be

$$S(1,\Omega) = \begin{pmatrix} B(1,\omega_1) & S_1(1,\omega_1) & \cdots & S_N(1,\omega_1) \\ \vdots & \vdots & & \vdots \\ B(1,\omega_K) & S_1(1,\omega_K) & \cdots & S_N(1,\omega_K) \end{pmatrix} \in \mathbb{R}^{K\times(N+1)}.$$

- Note that, together with $B(0)$ and $S(0) = (S_1(0), ..., S_N(0))^T$, $S(1, \Omega)$ fully characterizes the market model.
- One can also consider the matrix

$$S(0,\Omega) = \begin{pmatrix} B(0) & S_1(0) & \cdots & S_N(0) \\ \vdots & \vdots & & \vdots \\ B(0) & S_1(0) & \cdots & S_N(0) \end{pmatrix} \in \mathbb{R}^{K\times(N+1)},$$

  with the first row repeated K times.

Model specifications

- This way of specifying the market model emphasizes the linear algebra point of view on financial market models on finite probability spaces. That is:
  - Random variables are represented as elements in $\mathbb{R}^K$.
  - N random variables (or a N-dimensional random vector) are represented as elements in $\mathbb{R}^{K\times N}$.
  - Constants (degenerate random variables) can be represented as elements in $\mathbb{R}^K$ with all components being equal.
- We also consider the discounted payoff matrix $S^*(1,\Omega)$ in an obvious way.
- Note that $V(1), V^*(1), G, G^* \in \mathbb{R}^K$ associated to the trading strategy $H \in \mathbb{R}^{N+1}$ are given by

$$V(1) = S(1,\Omega)H, \qquad V^*(1) = S^*(1,\Omega)H,$$
$$G = \Delta S(\Omega)H, \quad \text{and} \quad G^* = \Delta S^*(\Omega)H,$$

where $\Delta S(\Omega) := S(1,\Omega) - S(0,\Omega)$, and $\Delta S^*(\Omega) := S^*(1,\Omega) - S^*(0,\Omega)$.

Model specifications

- A probability measure Q can also be seen as an element in $\mathbb{R}^K$.
- Q induces a linear functional on the set of random variables $\mathbb{E}_Q[\cdot] : \mathbb{R}^K \to \mathbb{R}$, called expectation under Q, given by

$$\mathbb{E}_Q[Z] = \sum_{k=1}^{K} Q(\omega_k) Z(\omega_k) = \sum_{k=1}^{K} Q_k Z_k = Q^T Z = Z^T Q.$$

- The expected value of the random vector of (discounted) assets $\overline{S}(1) := (B(1), S_1(1), ..., S_N(1))^T$ is given by

$$\mathbb{E}_Q\left[\overline{S}(1)\right] = S^T(1,\Omega) Q, \qquad \left(\mathbb{E}_Q\left[\overline{S}^*(1)\right] = S^{*T}(1,\Omega) Q, \right).$$

- Note also that one can write the expected values of $V(1)$ and $V^*(1)$as

$$\mathbb{E}_Q[V(1)] = H^T S^T(1,\Omega) Q = Q^T S(1,H) H,$$
$$\mathbb{E}_Q[V^*(1)] = H^T S^{*T}(1,\Omega) Q = Q^T S^*(1,H) H.$$

Model specifications

Example

- Consider $N = 1, K = 2$ $(\Omega = \{\omega_1, \omega_2\})$, $r = 1/9$, $B(0) = 1$, $B(1) = 1 + r = \frac{10}{9}$, $S_1(0) = 5$ and

$$S_1(1, \omega) = \begin{cases} \frac{20}{3} & \text{if} \quad \omega = \omega_1 \\ \frac{40}{9} & \text{if} \quad \omega = \omega_2 \end{cases} = \frac{20}{3}\mathbf{1}_{\{\omega_1\}}(\omega) + \frac{40}{9}\mathbf{1}_{\{\omega_2\}}(\omega).$$

- The previous notation for $S_1(1)$ emphasizes the random variable nature of $S_1(1)$.
- You can also see $S_1(1)$ as an element of $\mathbb{R}^K = \mathbb{R}^2$, i.e., a column vector $S_1(1) = \left(\frac{20}{3}, \frac{40}{9}\right)^T$.
- The discounted price process is given by $S_1^*(0) = S_1(0) / B(0) = 5/1 = 5$ and

$$S_1^*(1) = S_1(1) / B(1) = \left(\frac{\frac{20}{3}}{\frac{10}{9}}, \frac{\frac{40}{9}}{\frac{10}{9}}\right)^T = (6, 4)^T.$$

Model specifications

Example 1

- Next consider a trading strategy $H = (H_0, H_1)^T$.
  - At $t = 0$: we have

$$V(0) = H_0 B(0) + H_1 S_1(0) = H_0 + H_1 5,$$
$$V^*(0) = H_0 + H_1 S_1^*(0) = H_0 + H_1 5.$$

  - At $t = 1$: we have

$$\begin{aligned} V(1) &= H_0 B(1) + H_1 S_1(1) = \frac{10}{9} H_0 + H_1 S_1(1) \\ &= \begin{cases} \frac{10}{9} H_0 + \frac{20}{3} H_1 & \text{if} \quad \omega = \omega_1 \\ \frac{10}{9} H_0 + \frac{40}{9} H_1 & \text{if} \quad \omega = \omega_2 \end{cases}, \end{aligned}$$

$$\begin{aligned} V^*(1) &= H_0 + H_1 S_1^*(1) \\ &= \begin{cases} H_0 + 6 H_1 & \text{if} \quad \omega = \omega_1 \\ H_0 + 4 H_1 & \text{if} \quad \omega = \omega_2 \end{cases}, \end{aligned}$$

Model specifications

Example 1

$$G = H_0 r + H_1 \Delta S_1 = \frac{1}{9} H_0 + H_1 \left(S_1(1) - S_1(0) \right)$$
$$= \begin{cases} \frac{1}{9} H_0 + \left(\frac{20}{3} - 5 \right) H_1 = \frac{1}{9} H_0 + \frac{5}{3} H_1 & \text{if} \quad \omega = \omega_1 \\ \frac{1}{9} H_0 + \left(\frac{40}{9} - 5 \right) H_1 = \frac{1}{9} H_0 - \frac{5}{9} H_1 & \text{if} \quad \omega = \omega_2 \end{cases},$$

$$G^* = H_1 \Delta S_1^* = H_1 \left(S_1^*(1) - S_1^*(0) \right)$$
$$= \begin{cases} H_1 (6-5) = H_1 & \text{if} \quad \omega = \omega_1 \\ H_1 (4-5) = -H_1 & \text{if} \quad \omega = \omega_2 \end{cases},$$

- Please note that $V(1) = V(0) + G$ and $V^*(1) = V^*(0) + G^*$.

Dominant trading strategies

Dominant trading strategies

The following statements are equivalent

1. $\exists$ ***DTS***.
2. $\exists$ a trading strategy satisfying

$$\begin{cases} V(0) = 0 \\ V(1,\omega) > 0, \qquad \forall \omega \in \Omega \end{cases}. \tag{8}$$

3. $\exists$ a trading strategy satisfying

$$\begin{cases} V(0) < 0 \\ V(1,\omega) \geq 0, \qquad \forall \omega \in \Omega \end{cases}. \tag{9}$$

Proof.

Smartboard. □

- If in 2. and/or 3. we change V by V^* the result still holds.

Dominant trading strategies

- The existence of a dominant trading strategy is also unsatisfactory because leads to "illogical" pricing.
- It is useful to interpret $V(1)$ as the payoff of a contingent claim (think of options) and $V(0)$ as the price of this claim.
- Assume that $\hat{H}$ dominates $\widetilde{H}$.
- Then, the prices $\widehat{V}(0)$ and $\widetilde{V}(0)$ coincide but the payoffs will satisfy

$$\widehat{V}(1,\omega) > \widetilde{V}(1,\omega), \qquad \omega \in \Omega.$$

- This clearly does not make sense as it provides a sure positive profit with zero initial investment by taking a long position in $\widehat{V}$ and a short position in $\widetilde{V}$.

Linear pricing measures

- The following concept is useful because it provides a “logical” pricing rule.

A ***linear pricing measure*** (**LPM**) is a non-negative vector $\pi = (\pi(\omega_1), ..., \pi(\omega_K))^T$ such that for every trading strategy $H = (H_0, H_1, ..., H_N)^T$ the following holds

$$V^*(0) = \sum_{\omega \in \Omega} \pi(\omega) V^*(1, \omega). \tag{10}$$

- Note that equation (10) can be written as

$$H_0 + \sum_{n=1}^{N} H_n S_n^*(0) = \sum_{\omega \in \Omega} \pi(\omega) \left(H_0 + \sum_{n=1}^{N} H_n S_n^*(1, \omega) \right). \tag{11}$$

Linear pricing measures

Linear pricing measures

1. Let π be a **LPM**. Then, π is a probability measure on $\Omega = \{\omega_1, ..., \omega_K\}$.
2. π is a **LPM** $\Leftrightarrow$ π is a probability measure satisfying

$$S_n^*(0) = \sum_{\omega \in \Omega} S_n^*(1, \omega)\, \pi(\omega) =: \mathbb{E}_\pi [S_n^*(1)], \qquad n = 1, ..., N. \tag{12}$$

Proof.

Smartboard. □

Linear pricing measures

- The previous result says that

$$S_n^*(0) = \mathbb{E}_\pi [S_n^*(1)], \qquad n = 1, ..., N, \tag{13}$$
$$V^*(0) = \mathbb{E}_\pi [V^*(1)]. \tag{14}$$

- That is, the price/value at time 0 of a security can be obtained by taking expectations under a **LPM** π of the discounted terminal price/value of the security.
- In this context, equations (13) and (14) just say that the discounted processes S_n^* and V_n^* are martingales under π.
- Using a **LPM** each contingent claim $V(1, \omega)$ has a unique price and a claim that pays more than other for every $\omega \in \Omega$ will have a higher price (logical pricing).

Linear pricing measures and dominant trading strategies

$\exists$ **LPM** $\Longleftrightarrow \nexists$ **DTS**.

Proof.

Smartboard. □

- Financial market models allowing for **DTS** are not reasonable.
- But even less reasonable are models allowing for the failure of of the law of one price.

Law of one price

Law of one price

We say that the **law of one price** (**LOP**) holds for a financial market model if there do not exist two trading strategies $\widehat{H}$ and $\widetilde{H}$ such that

$$\begin{cases} \widehat{V}(0) > \widetilde{V}(0) \\ \widehat{V}(1,\omega) = \widetilde{V}(1,\omega), \qquad \forall \omega \in \Omega \end{cases}. \tag{15}$$

1. If in (15) we use $\widehat{V}^*$ and $\widetilde{V}^*$ we get the same concept.
2. **LOP** holds $\implies$ No ambiguity regarding the price at $t = 0$ ($V(0)$) of contingent claims ($V(1)$).
3. $\nexists$ two distinct trading strategies yielding the same payoff at $t = 1 \implies$ **LOP** holds.
4. **LOP** does not hold $\implies \exists$ two distinct trading strategies with the same final value but different initial value.

Law of one price and dominant trading strategies

$\nexists$ **DTS** $\Rightarrow$ **LOP** holds.

Proof.

- Suppose **LOP** does not hold. Then, there exist $\widehat{H}, \widetilde{H}$ such that $\widehat{V}^*(0) > \widetilde{V}^*(0)$ and $\widehat{V}^*(1) = \widetilde{V}^*(1)$.
- Since $\widehat{V}^*(1) = \widehat{V}^*(0) + \widehat{G}^*$ and $\widetilde{V}^*(1) = \widetilde{V}^*(0) + \widetilde{G}^*$, we have that $\widehat{G}^* < \widetilde{G}^*$.
- Define a new trading strategy H by setting $H_0 = -\sum_{n=1}^N H_n S_n^*(0)$, and $H_n = \widetilde{H}_n - \widehat{H}_n$, $n = 1, ..., N$.
- Then, $V^*(0) = H_0 + \sum_{n=1}^N H_n S_n^*(0) = 0$,

$$V^*(1) = V^*(0) + \sum_{n=1}^{N} \left(\widetilde{H}_n - \widehat{H}_n \right) \Delta S_n^* = \widetilde{G}^* - \widehat{G}^* > 0,$$

and by Lemma 20 there exists a **DTS**.

□

Law of one price and dominant trading strategies

1. **LOP** holds $\nexists$ **DTS**. That is, the converse of the previous lemma does not hold. It is possible to have **DTS** and **LOP** still holds.
2. If in a model $\exists$ **DTS** the situation is bad because it leads to illogical pricing and the existence of strategies with a sure positive final value with zero initial investment.
3. If in a model **LOP** does not hold the situation is even worse. It also allows for the existence of "**suicide strategies**", that is, strategies with positive initial investment and sure zero final value. Let $\widehat{H}, \widetilde{H}$ such that $\widehat{V}(0) > \widetilde{V}(0)$ and $\widehat{V}(1) = \widetilde{V}(1)$. Then, by the linearity of V with respect to H, we have that $H := \widehat{H} - \widetilde{H}$ satisfies

$$V(0) = \widehat{V}(0) - \widetilde{V}(0) > 0 \qquad \text{and} \qquad V(1) = \widehat{V}(1) - \widetilde{V}(1) = 0.$$

Example LOP does not hold

Example

- Take $K = 2, N = 1, r = 1$,$B(0) = 1$, $B(1) = 2$,$S(0) = 10$ and

$$S(1,\omega) = \begin{cases} 12 & \text{if} \quad \omega = \omega_1 \\ 12 & \text{if} \quad \omega = \omega_2 \end{cases} .$$

  That is, $S(1)$ is constant.

- Then,

$$\begin{aligned} V(0) &= H_0 B(0) + H_1 S(0) = H_0 + 10H_1, \\ V(1) &= H_0 B(1) + H_1 S(1) = 2H_0 + 12H_1. \end{aligned} \tag{16}$$

  Note that $V(1,\omega)$ is also constant.

- The previous linear system has a unique solution given by

$$H_0 = \frac{5}{4} V(1) - \frac{3}{2} V(0), \qquad H_1 = \frac{1}{4} V(0) - \frac{1}{8} V(1).$$

Example LOP does not hold

Example 2

- This means that, for fixed $V(1)$, there are an infinite number of strategies (each starting with a different $V(0)$) which yield $V(1) \implies$ **LOP** does not hold.
- In the same model, suppose now that $S(1,\omega_2) = 8$.
- Now, in addition to (16) we have

$$\left. \begin{array}{c} V(1,\omega_1) = H_0 B(1) + H_1 S(1,\omega_1) = 2H_0 + 12H_1, \\ V(1,\omega_2) = H_0 B(1) + H_1 S(1,\omega_2) = 2H_0 + 8H_1. \end{array} \right\} \tag{17}$$

- For arbitrary $V(1,\omega_1)$ and $V(1,\omega_2)$ the system (17) has a unique solution and taking into account (16) we have that $V(0)$ is uniquely determined $\implies$ **LOP** holds.

Example LOP does not hold

Example 2

- However, for $H = (H_0, H_1)^T = (10, -1)^T$ we have

$$\begin{aligned} V(0) &= H_0 + 10H_1 = 10 - 10 = 0, \\ V(1, \omega_1) &= 2H_0 + 12H_1 = 20 - 12 = 8 > 0, \\ V(1, \omega_2) &= 2H_0 + 12H_1 = 20 - 8 = 12 > 0. \end{aligned}$$

- Hence, H is a **DTS**.

Arbitrage opportunity

An ***arbitrage opportunity*** (**AO**) is a trading strategy satisfying:

a) $V(0) = 0.$

b) $V(1,\omega) \geq 0, \quad \omega \in \Omega.$

c) $\mathbb{E}[V(1)] > 0.$

1. c) can be changed by
   c') $\exists\, \omega \in \Omega$ such that $V(1,\omega) > 0.$
2. a), b) c) $\Longleftrightarrow V^*(0) = 0, V^*(1) \geq 0$, and $\mathbb{E}[V^*(1)] > 0.$
3. An **AO** is a trading strategy
   - with zero initial investment,
   - without the possibility of bearing a loss
   - with a strictly positive profit for at least one of the possible states of the economy.

Arbitrage opportunity

1. $\exists$ **DTS** $\implies$ $\exists$ **AO.**
2. $\exists$ **AO** $\exists$ **DTS.**

Proof.

1. By Lemma 20, we know that
   $\exists$ of **DTS** $\iff$ $\exists$ of H such that $V(0) = 0$ and $V(1, \omega) > 0, \omega \in \Omega$.
   But, if $V(1, \omega) > 0, \omega \in \Omega$ then
   $$\mathbb{E}[V(1)] = \sum_{\omega \in \Omega} V(1, \omega) P(\omega) > 0.$$
2. The following example provides a counterexample.

□

Arbitrage opportunity

Arbitrage opportunity

Example

- Take $K = 2, N = 1, r = 0, B(0) = 1, B(1) = 1, S(0) = S^*(0) = 10$ and

$$S(1,\omega) = S^*(1,\omega) = \begin{cases} 12 & \text{if} \quad \omega = \omega_1 \\ 10 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- Consider the trading strategy $H = (H_0, H_1)^T = (-10, 1)^T$, then $V(0) = H_0 B(0) + H_1 S(0) = -10 + 10 = 0$, and

$$V(1) = H_0 B(1) + H_1 S(1) = \begin{cases} -10 + 12 = 2 & \text{if} \quad \omega = \omega_1 \\ -10 + 10 = 0 & \text{if} \quad \omega = \omega_2 \end{cases}.$$

- Hence, H is an arbitrage opportunity.

Arbitrage opportunity

Example 3

- By Lemma 26 we know that the model does not contain **DTS** if and only if $\exists$ **LPM**.
- A **LPM** $\pi = (\pi_1, \pi_2)^T$ must satisfy $\pi \geq 0$ and

$$10 = S^*(0) = \mathbb{E}_\pi [S^*(1)] = 12\pi_1 + 10\pi_2.$$

- Hence, $\pi = (0, 1)^T$ is a **LPM** and we can conclude.

Arbitrage opportunity

H is an **AO** $\Longleftrightarrow G^* (\omega) \geq 0, \omega \in \Omega$ and $\mathbb{E} [G^*] > 0$.

Thank you!