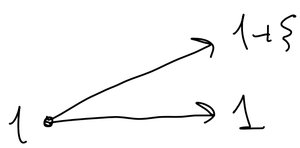
SOLUTIONS (SUGGESTED) TO THE TRIAL EXAM 2023

(1)



, $r = \xi/2$.

a) X pays 1 if stock goes up, zero otherwise.

Let a be # of stocks and b be the position in bank at time 0. To replicate X, we must have that

$$b\left(1+\frac{\xi}{2}\right) + a\cdot(1+\xi) = 1$$
$$b\left(1+\frac{\xi}{2}\right) + a \qquad = 0$$

$$\Rightarrow \quad a = -b\left(1+\frac{\xi}{2}\right) \Rightarrow -a + a + a\xi = 1$$

$$\Rightarrow \quad \underline{\underline{a = 1/\xi}}, \quad \underline{\underline{b}} = -\frac{1/\xi}{1+\xi/2} = \underline{\underline{-\frac{2}{2\xi+\xi^2}}}$$

We assume that $\xi > 0$!

b) The replicating portfolio has the initial cost

$$\underline{V_0} = b + a \cdot 1 = -\frac{2}{(2+\xi)\xi} + \frac{1}{\xi} = \underline{\frac{1}{2+\xi}}$$

If price C_0 of claim X is not V_0, we find an arbitrage possibility;

$\underline{V_0 > C_0}$ ie, claim is cheap, so buy it!

<u>At time 0</u>

Buy claim	$-\ C_0$
Sell hedge (go short!)	$+\ V_0$
Deposit in bank	$-\ (V_0 - C_0)$
	0

<u>At time 1</u>

Receive money from X	$\begin{cases} +1 \\ 0 \end{cases}$
Settle hedge	$\begin{cases} -1 \\ 0 \end{cases}$
Withdraw deposit	$+\ (V_0 - C_0)(1 + \frac{\xi}{2})$

$$+ (V_0 - C_0)(1+\tfrac{\xi}{2}) > 0$$

$\Rightarrow$ Arbitrage!

$V_0 < C_0$, i.e., claim is expensive, so sell it!

At time 0

Sell claim	$+\ C_0$
Buy hedge	$-\ V_0$
Deposit money in bank	$-\ (C_0 - V_0)$
	0

At time 1

Pay money to claim	$\begin{cases} -1 \\ 0 \end{cases}$
Sell hedge	$\begin{cases} +1 \\ 0 \end{cases}$
withdraw money in bank	$+ (C_0 - V_0)(1+\tfrac{\xi}{2})$
	$+ (C_0 - V_0)(1+\tfrac{\xi}{2}) > 0$

$\Rightarrow$ Arbitrage!

c)

Risk-neutral probability Q is such that discounted stock price is martingale: In one-period market, this means

$$1 = S(0) = E_Q[S^*(1)] = E_Q[S(1)] / (1+\tfrac{\xi}{2})$$

$$= \frac{1}{1+\frac{\xi}{2}} \left(q \cdot (1+\xi) + (1-q) \cdot 1 \right)$$

$$\Rightarrow \quad 1 + \frac{\xi}{2} = q\xi + 1 \quad \Rightarrow \quad \underline{\underline{q = \frac{1}{2}}}.$$

q is the risk-neutral probability for moving up.

$$E_Q[\underline{X}] = q \cdot 1 + (1-q) \cdot 0 = \frac{1}{2}.$$

$$\frac{1}{1+\frac{\xi}{2}} E_Q[\underline{X}] = \underline{\underline{\frac{1}{2+\xi}}} = V_0.$$

2

a) An algebra $\mathcal{F}$ is a collection of subsets of Ω which is closed under complement and union:

(i) $\phi \in \mathcal{F}$

(ii) If $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) If $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cup A_2 \in \mathcal{F}$

A filtration $\mathbb{F}$ is a family of algebras $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$, such that

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$
- $\mathcal{F}_s \subseteq \mathcal{F}_t$ when $s \leq t$
- $\mathcal{F}_T =$ all subsets of Ω.

b) $\mathcal{F}_0$ and $\mathcal{F}_2$ are trivially algebras, and we also have $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$. Show that $\mathcal{F}_1$ is an algebra:

(i) $\emptyset \in \mathcal{F}_1$ ✓

(ii) $\{\omega_3, \omega_4\}^c = \{\omega_1, \omega_2, \omega_5\} \in \mathcal{F}_1$

$\{\omega_1, \omega_2, \omega_5\}^c = \{\omega_3, \omega_4\} \in \mathcal{F}_1$ ✓

(iii) $\{\omega_3, \omega_4\} \cup \{\omega_1, \omega_2, \omega_5\} = \Omega \in \mathcal{F}_1$ ✓

$\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ is a filtration.

c) We check what sets Ψ is constant on:

$$\{\omega \in \Omega \mid \Psi(\omega) = 2\} = \{\omega_1, \omega_2\} \notin \mathcal{F}_1.$$

$\Rightarrow \Psi$ is not $\mathcal{F}_1$-measurable.

It is strictly speaking not necessary, but we could also check that

$$\{\omega \in \Omega \mid \Psi(\omega) = 1\} = \{\omega_3, \omega_4, \omega_5\} \notin \mathcal{F}_1$$

Partition of $\mathcal{F}_1$: Let $A_1 = \{\omega_1, \omega_2, \omega_5\}$, $A_2 = \{\omega_3, \omega_4\}$. We note that $A_1, A_2 \in \mathcal{F}_1$, $A_1 \cup A_2 = \Omega$ and $A_1 \cap A_2 = \emptyset$. Hence, A_1 and A_2 is a partition of $\mathcal{F}_1$.

Let us compute the conditional expectation $E[\Psi \mid \mathcal{F}_1]$. By definition

$$E[\Psi \mid \mathcal{F}_1](\omega) = \begin{cases} E[\Psi \mid A_1], & \omega \in A_1 \\ E[\Psi \mid A_2], & \omega \in A_2. \end{cases}$$

$$E[\Psi \mid A_1] = \sum_{i=1}^{5} \Psi(\omega_i) \cdot P(\{\omega_i\} \mid A_1)$$

Bayes' Formula

$$= \sum_{i=1}^{5} \underline{X}(\omega_i) \frac{P(\{\omega_i\} \cap A_1)}{P(A_1)}$$

$$\{\omega_i\} \cap A_1 = \begin{cases} \{\omega_i\}, & i = 1,2,5 \\ \phi, & i = 3,4 \end{cases}$$

$$= \underline{X}(\omega_1) \frac{P(\{\omega_1\})}{P(A_1)} + \underline{X}(\omega_2) \frac{P(\{\omega_2\})}{P(A_1)} + \underline{X}(\omega_5) \frac{P(\{\omega_5\})}{P(A_1)}$$

$$P(A_1) = P(\{\omega_1\} \cup \{\omega_2\} \cup \{\omega_5\}) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$$

$$= 2 \cdot \frac{1/5}{3/5} + 2 \cdot \frac{1/5}{3/5} + 1 \cdot \frac{1/5}{3/5}$$

$$= \frac{5}{3}$$

$$E[\underline{X} | A_2] = \sum_{i=1}^{5} \underline{X}(\omega_i) \cdot \frac{P(\{\omega_i\} \cap A_2)}{P(A_2)}$$

$$\{\omega_i\} \cap A_2 = \begin{cases} \{\omega_i\}, & i = 3,4 \\ \phi, & i = 1,2,5 \end{cases}$$

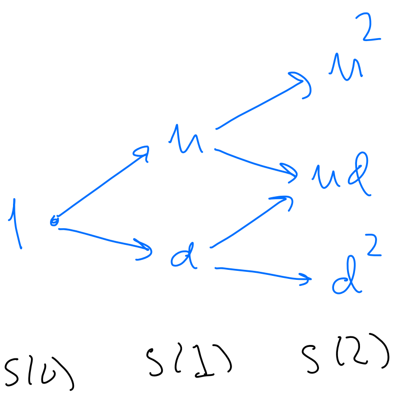
$$P(A_2) = P(\{\omega_3\} \cup \{\omega_4\}) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$= \underline{X}(\omega_3) \frac{P(\{\omega_3\})}{P(A_2)} + \underline{X}(\omega_4) \frac{P(\{\omega_4\})}{P(A_2)}$$

$$= 1 \cdot \frac{1/5}{2/5} + 1 \cdot \frac{1/5}{2/5} = \underline{\underline{1}}$$

$$E[Y \mid \mathcal{F}_1](\omega) = \begin{cases} 5/3, & \omega = \omega_1, \omega_2, \omega_5 \\ 1, & \omega = \omega_3, \omega_4 \end{cases}$$

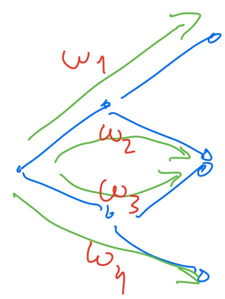
3



$r > 0, \quad 0 < d < 1+r < u.$

a)

ω_1 : stock goes up and up
ω_2 : –"– up –"– down
ω_3 : –"– down –"– up
ω_4 : –"– down –"– down.



$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}.$

$\mathcal{F}_1$ is generated by $S(0)$ and $S(1)$. Since $S(0) = 1$, we find only the sets Ω and $\emptyset$ from this, which

We know in any case are sets in $\mathcal{F}_1$.

$\{\omega \in \Omega \mid S(\omega) = u\} = \{\omega_1, \omega_2\}$
$\{\omega \in \Omega \mid S(\omega) = d\} = \{\omega_3, \omega_4\}$.

$\mathcal{F}_1$ must consist of the sets $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$. We see that since

- $\{\omega_1, \omega_2\}^c = \{\omega_3, \omega_4\} \in \mathcal{F}_1$
  $\{\omega_3, \omega_4\}^c = \{\omega_1, \omega_2\} \in \mathcal{F}_1$
- $\{\omega_1, \omega_2\} \cup \{\omega_3, \omega_4\} = \Omega \in \mathcal{F}_1$

$\mathcal{F}_1 = \{\phi, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$ is an algebra.

$\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_2 = \{\phi, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}, \{\omega_2, \omega_4\}, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_1, \omega_3, \omega_4\}, \{\omega_2, \omega_3, \omega_4\}, \Omega\}$. ie, all subsets of Ω!

b) By the general pricing formula;

$$C^*(t) = E_Q[X^* \mid \mathcal{F}_t], \quad t = 0, 1, 2.$$

$t=2$:

this is $\mathcal{F}_2$-measurable!

$$C^*(2) = E_Q[X^* \mid \mathcal{F}_2] = X^*$$ (by properties of cond. expectation!)

$$\Leftrightarrow \frac{C(2)}{(1+r)^2} = \frac{X}{(1+r)^2} \Leftrightarrow C(2) = X = \begin{cases} 0, & \omega = \omega_1, \omega_2, \omega_3 \\ 1, & \omega = \omega_4 \end{cases}$$

(this we could also show by no-arbitrage arguments!)

$t=1$

$$C^*(1) = E_Q[X^* \mid \mathcal{F}_1]$$

$$\Leftrightarrow \frac{C(1)}{1+r} = E_Q\left[\frac{X}{(1+r)^2} \mid \mathcal{F}_1\right]$$

$$\Leftrightarrow C(1) = \frac{1}{1+r} E_Q[X \mid \mathcal{F}_1].$$

Let us calculate the conditional expectation:

Notice that $A_1 = \{\omega_1, \omega_2\}$ and $A_2 = \{\omega_3, \omega_4\}$ is a partition of $\mathcal{F}_1$ since $A_1 \cup A_2 = \Omega$ and $A_1 \cap A_2 = \emptyset$, and obviously $A_1, A_2 \in \mathcal{F}_1$.

We have $Q(A_1) = Q(\{\omega_1\} \cup \{\omega_2\}) = q^2 + q(1-q) = q$

and $Q(A_2) = Q(\{\omega_3\} \cup \{\omega_4\}) = (1-q)q + (1-q)^2 = 1-q$

$$\{\omega_i\} \cap A_1 = \begin{cases} \{\omega_i\}, & i=1,2 \\ \phi, & i=3,4 \end{cases} \qquad \{\omega_i\} \cap A_2 = \begin{cases} \{\omega_i\}, & i=3,4 \\ \phi, & i=1,2 \end{cases}$$

We find

$$E_Q[X \mid A_1] = \sum_{i=1}^{4} X(\omega_i) \frac{Q(\{\omega_i\} \cap A_1)}{Q(A_1)}$$

$$= \underbrace{X(\omega_1)}_{=0} \frac{Q(\{\omega_1\})}{Q(A_1)} + \underbrace{X(\omega_2)}_{=0} \frac{Q(\{\omega_2\})}{Q(A_1)}$$

$$= 0.$$

$$E_Q[X \mid A_2] = \underbrace{X(\omega_3)}_{=0} \frac{Q(\{\omega_3\})}{Q(A_2)} + \underbrace{X(\omega_4)}_{=1} \frac{Q(\{\omega_4\})}{Q(A_2)}$$

$$= 1 \cdot \frac{Q(\{\omega_4\})}{Q(A_2)} = \frac{(1-q)^2}{1-q} = 1-q.$$

$$E_Q[X \mid F_1](\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ 1-q, & \omega = \omega_3, \omega_4. \end{cases}$$

$$C(1)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ \frac{1-q}{1+r}, & \omega = \omega_3, \omega_4 \end{cases}$$

$\underline{t=0}$ $C^*(0) = E_Q[X^* \mid \mathcal{F}_0] = E_Q[X^*]$

$$\Rightarrow \underline{\underline{C(0)}} = E_Q\left[\frac{X}{(1+r)^2}\right] = \frac{1}{(1+r)^2} \cdot E_Q[X]$$

$$= \frac{1}{(1+r)^2}\left(0 \cdot q^2 + 0 \cdot q(1-q) + 0 \cdot (1-q)q + 1 \cdot (1-q)^2\right)$$

$$= \underline{\underline{\frac{(1-q)^2}{(1+r)^2}}}$$

We can now spell out what q is, to find that $1-q = \dfrac{u-(1+r)}{u-d}$.

Thus

$$\underline{\underline{C(0) = \frac{(u-(1+r))^2}{(u-d)^2(1+r)^2}}}$$

$$C(1)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ \dfrac{u-(1+r)}{(u-d)(1+r)}, & \omega = \omega_3, \omega_4 \end{cases}$$

$$C(2)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2, \omega_3 \\ 1, & \omega = \omega_4 \end{cases}$$

c) We start with creating a portfolio at time 1 which is so that we replicate the claim at time 2.

$H_1(2)$ is the investment in the asset done at time 1, while $H_0(2)$ is the bank position. Both $H_1(2)$ and $H_0(2)$ are $\mathcal{F}_1$-measurable random variables. This means that $H_1(2)$ and $H_0(2)$ are constants on $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$.

$$\Rightarrow \quad \begin{aligned} H_1(2)(\omega_1) &= H_1(2)(\omega_2) \\ H_0(2)(\omega_1) &= H_0(2)(\omega_2) \\ H_1(2)(\omega_3) &= H_1(2)(\omega_4) \\ H_0(2)(\omega_3) &= H_0(2)(\omega_4) \end{aligned}$$

To be replicating, we must have

$$H_0(2)(\omega)(1+r) + H_1(2)(\omega)S(2)(\omega) = X(\omega)$$

for $\omega = \omega_1, \omega_2, \omega_3, \omega_4$.

Or,

$$\begin{cases} H_0(2)(\omega_1)(1+r) + H_1(2)(\omega_1)u^2 = 0 \end{cases}$$

$$\mathrm{I}\quad \left\{ H_0(2)(\omega_1)(1+r) + H_1(2)(\omega_1)\, u d = 0 \right.$$

$H_0(2)(\omega_1) = H_0(2)(\omega_2)$, $H_1(2)(\omega_1) = H_1(2)(\omega_2)$

$$\mathrm{II}\quad \begin{cases} H_0(2)(\omega_3)(1+r) + H_1(2)(\omega_3)\, u d = 0 \\ H_0(2)(\omega_3)(1+r) + H_1(2)(\omega_3)\, d^2 = 1 \end{cases}$$

I is a system of 2 equations with 2 unknowns, $H_0(2)(\omega_1)$ and $H_1(2)(\omega_1)$. The solution is obviously

$$\underline{H_0(2)(\omega_1) = H_1(2)(\omega_1) = 0.}$$

II is also a 2×2 system of equations, and we solve to find

$$H_1(2)(\omega_3)(ud - d^2) = -1 \Rightarrow \underline{H_1(2)(\omega_3) = -\frac{1}{d(u-d)}}$$

$$\Rightarrow \underline{H_0(2)(\omega_3) = \frac{u}{(1+r)(u-d)}}$$

This portfolio has the value at time 1 given by

$$V_1(\omega) = H_0(2)(\omega) + H_1(2)(\omega)\, S(1)(\omega)$$

or

$$V_1(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ \dfrac{u-(1+r)}{(1+r)(u-d)}, & \omega = \omega_3, \omega_4. \end{cases}$$

Next, we find a position in the bank $H_0(1)$ and in the asset $H_1(1)$, taken at time 0, such that we replicate V_1 (this ensures self-financing!) i.e.,

$$H_0(1)(1+r) + H_1(1)\cdot S(1)(\omega) = V_1(\omega)$$

or

$$\begin{cases} H_0(1)(1+r) + H_1(1)\, u = 0 \\ H_0(1)(1+r) + H_1(1)\, d = \dfrac{u-(1+r)}{(1+r)(u-d)} \end{cases}$$

Again we have a 2x2 system, where we find

$$H_1(1)(u-d) = -\frac{u-(1+r)}{(1+r)(u-d)} \Rightarrow \underline{H_1(1)} = \underline{-\frac{u-(1+r)}{(1+r)(u-d)^2}}$$

$$\Rightarrow H_0(1) = \frac{u(u-(1+r))}{(1+r)^2(u-d)^2}$$

The replicating strategy is therefore

$$H_0(1) = \frac{u(u-(1+r))}{(1+r)^2(u-d)^2}, \quad H_0(2)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ \dfrac{u}{(1+r)(u-d)}, & \omega = \omega_3, \omega_4 \end{cases}$$

$$H_1(1) = -\frac{u-(1+r)}{(1+r)(u-d)^2}, \quad H_1(2)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ -\frac{1}{d(u-d)}, & \omega = \omega_3, \omega_4 \end{cases}$$

By no-arbitrage, we must have that $V_0 = C(0)$, $V_1 = C(1)$, and $V_2 = C(2) = \underline{X}$.

Indeed, $V_2 = \underline{X}$ is by construction of the replicating portfolio. We also see above that $V_1 = C(1)$, so we have a sanity check of our calculations.

We can also check our calculations of the replicating portfolio by noting that

$$\begin{aligned} \underline{V_0} &= H_0(1) + H_1(1)S(0) \\ &= \frac{u(u-(1+r))}{(1+r)^2(u-d)^2} - \frac{u-(1+r)}{(1+r)(u-d)^2} \\ &= \frac{(u-(1+r))^2}{(1+r)^2(u-d)^2} = \underline{C(0)} \; ! \end{aligned}$$

4

a) Recall from lectures, learn by heart for the exam!!

b) The price, call it $S(t)$ for "straddle", of a long call and put position, is

$$\underline{\underline{S(t)}} = \underset{\substack{\uparrow \\ \text{price of call}}}{C(t)} + \underset{\substack{\uparrow \\ \text{price of put}}}{P(t)}$$

put-call parity

$$= C(t) + \left(C(t) - S(t) + Ke^{-rT}\right)$$

$$= \underline{\underline{2C(t) - S(t) + Ke^{-rT}}}$$

5

a) Suppose $\underline{F_t(T) > S(t)(1+r)^{T-t}}$.

Then we sell forward, and buy the asset on spot at time t :

Time t :

Sell forward	0
Buy spot	$-S(t)$
Finance this by borrowing money	$+S(t)$
	0

Time T

Deliver asset, which you have sold forwards, receive forward price	$+F_t(T)$
Settle the loan	$-S(t)(1+r)^{T-t}$
	$F_t(T)-S(t)(1+r)^{T-t} > 0.$

Sure profit from zero investment $\Rightarrow$ Arbitrage opportunity.

Suppose $\underline{F_t(T) < S(t)(1+r)^{T-t}}$

Time t

Buy forward	0
Short asset	$+S(t)$

Deposit money in bank	$-S(t)$
	0

Time T

Receive the asset from buying forward, pay forward price	$-F_t(T)$
Settle short position with asset from forward contract	0
Withdraw money from bank	$+S(t)(1+r)^{T-t}$
	$S(t)(1+r)^{T-t} - F_t(T) > 0$

Again, sure profit from zero investment $\Rightarrow$ Arbitrage opportunity.

Unless $F_t(T) = S(t)(1+r)^{T-t}$ we have an arbitrage opportunity.

b) By definition of a risk-neutral probability Q,

(i) $Q(\omega) > 0 \quad \forall\, \omega \in \Omega$

(ii) $S^*(t) := S(t)/(1+r)^t$ is a Q-martingale.

The last point means that $S(t)$ is $\mathcal{F}_t$-measurable for all $t=0,\dots,T$ (or, $(S(t))_{t=0}^{T}$ is $\mathbb{F}$-adapted)

and $$E_Q[S^*(s) \mid \mathcal{F}_t] = S^*(t), \quad s \geq t.$$

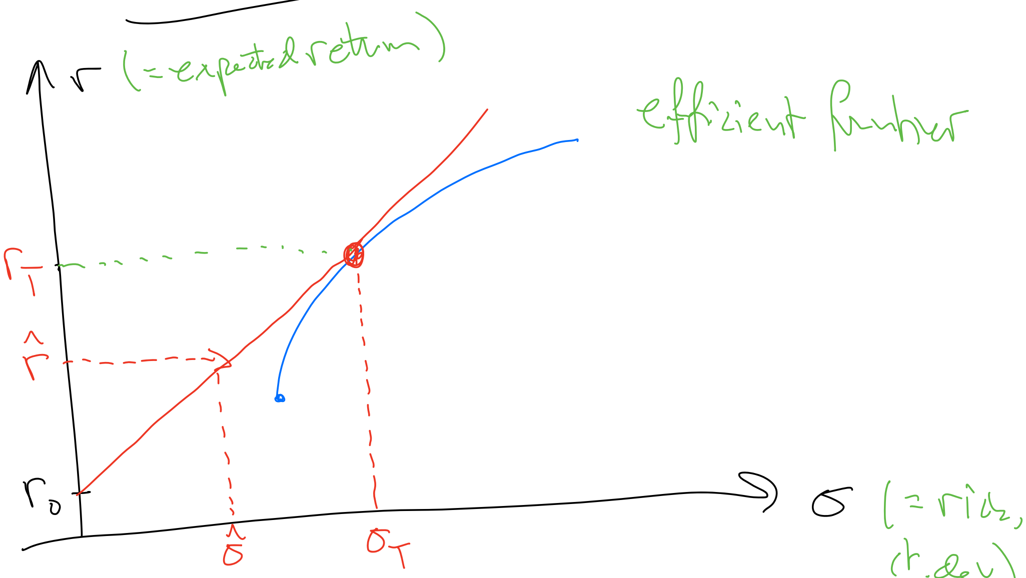
By this last property, we see that

$$E_Q\left[\frac{S(T)}{(1+r)^T} \,\Big|\, \mathcal{F}_t\right] = \frac{S(t)}{(1+r)^t} \quad \text{when}$$

letting $s=T$.

or $$E_Q[S(T) \mid \mathcal{F}_t] = S(t)(1+r)^{T-t} = F_t(T)$$

6



r_0 is the risk-free interest rate

The tangent portfolio is the portfolio on the efficient frontier which is tangential to the frontier and the tangent passes through r_0 on the r-axis. We can have a portfolio with any risk $\hat{\sigma}$, with $0 \le \hat{\sigma} \le \sigma_T$, by mixing a bank deposit with the tangent portfolio. The bank deposit has risk-return $(0, r_0)$, while tangent portfolio has (σ_T, r_T).

Choose x so that $x r_0 + (1-x) r_T = \hat{r}$, where $\hat{r}$ is the desired return. This means

$$x = \frac{r_T - \hat{r}}{r_T - r_0}$$ invested in the bank.

Risk is then $\hat{\sigma}^2 = (1-x)^2 \sigma_T^2$.