STK-MAT3710: Solution to mandatory assignment. Fall 2019

Problem 1. When we multiply out the product, we get

$$E\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} E\left[X_iX_jX_kX_l\right]$$

Let us take a closer look at the terms $E\left[X_iX_jX_kX_l\right]$. If one of the factors X_i, X_j, X_k, X_l (let us say X_i for simplicity) is different from the others, we get $E\left[X_iX_jX_kX_l\right] = E\left[X_i\right]E\left[X_jX_kX_l\right] = 0$ by independence. Hence the only contributions to the sum are from terms $E\left[X_iX_jX_kX_l\right]$ where none of the factors X_i, X_j, X_k, X_l are different from all the others. This leaves only two possibilities; either all four are equal (i.e. $X_i = X_j = X_k = X_l$) or they come in groups of two, e.g, $X_i = X_j$ and $X_k = X_l$. If we fix a number r, the term $E[X_r^4]$ only occurs once in the big sum $\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} E\left[X_iX_jX_kX_l\right]$ above (we need all four indices i, j, k, l to be equal to r), but if we fix two different numbers p and q, with $p < q$, the term $E\left[X_p^2X_q^2\right]$ arises in $6 = \binom{4}{2}$ different ways:

(i) $i = j = p$ and $k = l = q$

(ii) $i = k = p$ and $j = l = q$

(iii) $i = l = p$ and $j = k = q$

(iv) $i = j = q$ and $k = l = p$

(v) $i = k = q$ and $k = l = p$

(vi) $i = l = q$ and $j = k = q$

Summing up the nonzero terms, we get

$$E\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = \sum_{r=1}^{n} E\left[X_r^4\right] + 6\sum_{q=1}^{n}\sum_{p=1}^{q-1} E\left[X_p^2\right]E\left[X_q^2\right]$$

Remark: To be absolutely correct, we should perhaps insert a proof that all combinations $X_iX_jX_kX_l$ are integrable. This follows from Theorem 3.5 and Lyapounov's inequality (Corollary 3.23b) in the textbook.

Problem 2 a) By one of De Morgan's laws, we have

$$(A^c \cup B)^c \overset{\text{De M.}}{=} ((A^c)^c \cap B^c = A \cap B^c = A \setminus B$$

b) Observe first that if C, D are two disjoint sets in $\mathcal{D}$, then $C \cup D \in \mathcal{D}$ since

$$C \cup D = C \cup D \cup \emptyset \cup \emptyset \cup \ldots$$

is a countable, disjoint union of sets in $\mathcal{D}$.

We are now ready for the problem. According to part a), $A \setminus B = (A^c \cup B)^c$, and since $B \subseteq A$, the union is disjoint. By the observation above, $A^c \cup B \in \mathcal{D}$, and thus $A \setminus B = (A^c \cup B)^c \in \mathcal{D}$ by (ii).

c) Assume that $\{A_n\}$ is an increasing sequence of sets in $\mathcal{D}$, and define

$$B_1 = A_1, \quad B_2 = A_2 \setminus A_1, \quad B_3 = A_3 \setminus A_2 \quad \text{etc.}$$

According to b), we have $B_n \in \mathcal{D}$, and since the B_n's are disjoint, we $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{D}$ from (iii). By construction, $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$, and hence $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

d) The are many possibilities, but one is to put $\Omega = \{1, 2, 3, 4\}$ and let

$$\mathcal{D} = \{A \subseteq \Omega \ : \ A \text{ has an even number of elements}\}$$

Then $\mathcal{D}$ is easily seen to satisfy (i)-(iii), but $\mathcal{D}$ is *not* a σ-algebra as $\{1, 2\}$ and $\{2, 3\}$ are both in $\mathcal{D}$, but their union $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$ is not.

e) We have to check the three axioms for σ-algebras:

(i) $\emptyset \in \mathcal{D}$

(ii) If $A \in \mathcal{D}$, then $A^c \in \mathcal{D}$.

(iii) If $\{A_n\}$ is a sequence of sets in $\mathcal{D}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

The first two are automatically satisfied since $\mathcal{D}$ is a D-system. To check (iii), first observe that since $\mathcal{D}$ is closed under finite intersections, it is also closed under finite unions as $A \cup B = (A^c \cap B^c)^c$ by De Morgan. This means that if we define a new sequence $\{B_n\}$ by $B_n = A_1 \cup A_2 \cup \ldots \cup A_n$, we have an increasing sequence $\{B_n\}$ of sets in $\mathcal{D}$. By c), the union $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{D}$, and since $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$, we have proved that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

Problem 3. a) Assume first that $I \cap J = \emptyset$. If n is the length of $C_{I,\alpha}$ and k is the length of $C_{J,\beta}$, then $C_{I,\alpha} \cap C_{J,\beta}$ is a cylinder set of length $n + k$. Hence

$$P(C_{I,\alpha} \cap C_{J,\beta}) = \frac{1}{2^{n+k}} = \frac{1}{2^n} \cdot \frac{1}{2^k} = P(C_{I,\alpha})P(C_{J,\beta})$$

which means that $C_{I,\alpha}$ and $C_{J,\beta}$ are independent.

For the case $I \cap J \neq \emptyset$, there are two possibilities. If $C_{I,\alpha}$ and $C_{J,\beta}$ contradict each other on an element in $I \cap J$, then $C_{I,\alpha} \cap C_{J,\beta} = \emptyset$, and hence $P(C_{I,\alpha} \cap C_{J,\beta}) = 0$. As $P(C_{I,\alpha})P(C_{J,\beta}) \neq 0$, this proves that $C_{I,\alpha}$ and $C_{J,\beta}$ are dependent in this case. The other possibility is that $C_{I,\alpha}$ and $C_{J,\beta}$ agree on all elements in $I \cap J$. If n is the length of $C_{I,\alpha}$, k is the length of $C_{J,\beta}$, and $m = |I \cap J|$ is the size of the overlap, then $C_{I,\alpha} \cap C_{J,\beta}$ is a cylinder set of length $n + k - m$. Hence

$$P(C_{I,\alpha} \cap C_{J,\beta}) = \frac{1}{2^{n+k-m}} \neq \frac{1}{2^n} \cdot \frac{1}{2^k} = P(C_{I,\alpha})P(C_{J,\beta})$$

showing that $C_{I,\alpha}$ and $C_{J,\beta}$ is dependent also in this case.

b) There are many ways to argue, but here is one that looks forward to the next part of the problem. Let $A_n = \{\omega \in \Omega : \omega_n = H\}$. Then the sets $\{A_n\}$ are independent (use the same argument as in part a)), and since $P(A_n) = \frac{1}{2}$, we clearly have $\sum_{n=1}^{\infty} P(A_n) = \infty$. By part (ii) of Borel-Cantelli's lemma, this means that $P[\limsup_n A_n] = 1$. As $\omega \in \limsup_n A_n$ means that $\omega \in A_n$ (i.e. $\omega_n = H$) for infinitely many n, the assertion is proved.

c) We are going to use a slightly more sophisticated version of the argument in part b). We chop up $\mathbb{N}$ into sequences of length n: $I_1 = \{1, 2, \ldots, n\}$, $I_2 = \{n+1, n+2, \ldots, 2n\}$, $\ldots$, $I_k = \{(k-1)n+1, (k-1)n+2, \ldots, kn\}$, etc.

Let B_k be the set of all ω's such that the tuple α occurs on interval I_k in the sense that $\omega_{(k-1)n+1} = \alpha_1$, $\omega_{(k-1)n+2} = \alpha_2$ etc. Then the B_k's are independent and $P(B_k) = \frac{1}{2^n}$. Clearly, $\sum_{k=1}^{\infty} P(B_k) = \infty$, and by part (ii) of Borel-Cantelli's lemma, we have $P[\limsup_k B_k] = 1$. As $\omega \in \limsup_k B_k$ means that $\omega \in B_k$ for infinitely many k, the assertion is proved.

d) For each n-tuple α, let

$$\Omega_\alpha = \{\omega \in \Omega : \alpha \text{ occurs only finitely many times in } \omega\}$$

By part c), $P(\Omega_\alpha) = 0$. As there are only finitely many n-tuples of a given length n, the set

$$\Omega_n = \bigcup_{\text{length}(\alpha)=n} \Omega_\alpha$$

must also have probability 0, and so must

$$\Omega' = \bigcup_{n \in \mathbb{N}} \Omega_n$$

As

$$\omega \in \Omega' \iff \text{there is a tuple } \alpha \text{ which occurs only finitely many times in } \omega$$

the assertion is proved.