STK-MAT3710: Trial Exam 1, Fall 2019

Problem 1: The random variable X is given by: $P[X = 0] = \frac{1}{2}$, $P[X = 1] = P[X = -1] = \frac{1}{4}$.

a) Find a real expression for the characteristic function of X.

b) Assume that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of independent random variables with the same distribution as X. Find the characteristic function of

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \; .$$

c) Use the result in b) to show directly that S_n converges in distribution to a normal distribution ("show directly" means that you are not allowed to use a version of the Central Limit Theorem).

Problem 2. Let $\mathcal{T} = \{0, 1, 2, \ldots\}$ be a timeline and $\{\mathcal{F}_n\}_{n\in\mathcal{T}}$ a filtration. Assume that $\{X_n\}_{n\in\mathcal{T}}$ is an adapted, integrable process, and let $\Delta X_k = X_{k+1} - X_k$ be the forward increment of X at time k. Assume that $|\Delta X_k| < 1$ for all k.

a) Define a new process Y by putting $Y_0 = 1$ and

$$Y_n = \prod_{k=0}^{n-1} (1 + \Delta X_k) \; .$$

Show that if $\{X_n\}$ is a submartingale, so is $\{Y_n\}$.

b) Let $m_k = E[\Delta X_k]$. Show that if for all k the increment ΔX_k is independent of $\mathcal{F}_k$, then the process

$$Z_n = \frac{Y_n}{\prod_{k=0}^{n-1}(1 + m_k)} = \prod_{k=0}^{n-1} \frac{1 + \Delta X_k}{1 + m_k}$$

is a martingale.

Problem 3.

a) Let $\{X_n\}$ be a sequence of random variables converging in distribution to X, and let $\{Y_n\}$ be a sequence of random variables converging in distribution to Y. Show by example that $\{X_n + Y_n\}$ need not converge in distribution to $X + Y$.

b) Show that if $\{X_n\}$ converges in distribution to X, then $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)$ for all t, where ϕ_{X_n}, ϕ_X are the characteristic functions of X_n and X, respectively.

c) Assume that X_n and Y_n are independent for each n, and that $\{X_n\}$ and $\{Y_n\}$ converge in distribution to two independent random variables X and Y. Show that $\{X_n + Y_n\}$ converges in distribution to $X + Y$.

Problem 4: Let $\{x_i\}$ be a sequence of real numbers and let

$$s_n = \frac{x_1 + x_2 + \cdots + x_n}{\sqrt{n}} \ .$$

For $k \leq n$, put:

$$s_n^k = \frac{x_k + x_{k+1} + \cdots + x_n}{\sqrt{n}} \ .$$

a) Show that for all k, we have

$$\limsup_{n\to\infty} s_n^k = \limsup_{n\to\infty} s_n \ .$$

Assume now that $\{X_n\}$ is sequence of independent random variables. Let

$$S_n = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}} \ .$$

b) Let B be a Borel set. Show that

$$\Lambda = \{\omega \,:\, \limsup_{n\to\infty} S_n(\omega) \in B\}$$

is a tail event. Which values can $P(\Lambda)$ have?

THE END