STK-MAT3710: Trial Exam 1, Fall 2019: Solution

Problem 1: a) The characteristic function is

$$\phi_X(t) = E[e^{itX}] = e^{it\cdot 0}\frac{1}{2} + e^{it\cdot 1}\frac{1}{4} + e^{it(-1)}\frac{1}{4} = \frac{1}{2} + \frac{e^{it}+e^{-it}}{4} = \frac{1}{2}(1+\cos t)\ .$$

b) Using the independence, we have

$$\phi_{S_n}(t) = E\left[e^{itS_n}\right] = E\left[e^{i\frac{t}{\sqrt{n}}X_1}e^{i\frac{t}{\sqrt{n}}X_2}\cdot\ldots\cdot e^{i\frac{t}{\sqrt{n}}X_n}\right]$$

$$= E\left[e^{i\frac{t}{\sqrt{n}}X_1}\right]E\left[e^{i\frac{t}{\sqrt{n}}X_2}\right]\cdot\ldots\cdot E\left[e^{i\frac{t}{\sqrt{n}}X_n}\right] = \left(\phi(\frac{t}{\sqrt{n}})\right)^n = \frac{1}{2^n}\left(1+\cos(\frac{t}{\sqrt{n}})\right)^n\ .$$

c) The Taylor expansion for cosine is $\cos x = 1 - \frac{x^2}{2} + o(x^2)$, and hence

$$\phi_{S_n}(t) = \frac{1}{2^n}\left(1+\cos(\frac{t}{\sqrt{n}})\right)^n = \frac{1}{2^n}\left(1+1-\frac{t^2}{2n}+o(\frac{t^2}{n})\right)^n = \left(1-\frac{t^2}{4n}+o(\frac{t^2}{n})\right)^n\ .$$

Consequently (e.g. by Lemma 6.34)

$$\phi_{S_n}(t) = \left(1-\frac{t^2}{4n}+o\left(\frac{t^2}{n}\right)\right)^n \to e^{-\frac{t^2}{4}}.$$

As $e^{-\frac{t^2}{4}}$ is the characteristic function of a normal distribution with mean 0 and variance $\sigma^2 = \frac{1}{2}$, the result follows from Lévy's Continuity Theorem.

Problem 2. a) Y_n is clearly adapted, and since $|\Delta X_k| < 1$, we have $|Y_n| < 2^n$, and hence Y_n is integrable. To prove the submartingale property, note that

$$Y_{n+1} = \prod_{k=0}^{n}(1+\Delta X_k) = Y_n(1+\Delta X_n)\ .$$

As Y_n is $\mathcal{F}_n$-measurable, we get

$$E[Y_{n+1}|\mathcal{F}_n] = E\left[Y_n\left(1+\Delta X_n\right)|\mathcal{F}_n\right] = Y_n E\left[1+\Delta X_n|\mathcal{F}_n\right] = Y_n(1+E\left[\Delta X_n|\mathcal{F}_n\right])\ .$$

Note that since $\{X_n\}$ is a submartingale, $E[\Delta X_n|\mathcal{F}_n] \geq 0$, and hence $(1 + E\left[\Delta X_n|\mathcal{F}_n\right]) \geq 1$. Also, since $|\Delta X_k| < 1$, we have $Y_n > 0$. Thus

$$E[Y_{n+1}|\mathcal{F}_n] = Y_n(1+E\left[\Delta X_n|\mathcal{F}_n\right]) \geq Y_n\ ,$$

which shows that $\{Y_n\}$ is a submartingale.

b) If ΔX_n is independent of $\mathcal{F}_n$, then $E[\Delta X_n|\mathcal{F}_n] = E[\Delta X_n] = m_n$. Hence by calculations similar to those in b), we get

$$E[Z_{n+1}|\mathcal{F}_n] = E\left[Z_n\frac{1+\Delta X_n}{1+m_n}|\mathcal{F}_n\right] = \frac{Z_n}{1+m_n}E\left[1+\Delta X_n|\mathcal{F}_n\right] = Z_n\ ,$$

which shows that Z_n is a martingale.

Problem 3. a) Let X be a binomial random variable; i.e. $P[X=1]=P[X=-1]=\frac{1}{2}$. Put $X_n=Y_n=X$; then $X_n+Y_n=2X$ for all n, and $\{X_n+Y_n\}$ clearly converges in distribution to $2X$. Also, $\{X_n\}$ converges in distribution to X. Choose Y to be an *independent* copy of Y. Then $\{Y_n\}$ converges in distribution to Y (as they all have the same distribution), but as we have already seen, $\{X_n+Y_n\}$ converges to $2X$ in distribution, and not to $X+Y$ (these distributions are not the same as $P[2X=2]=P[2X=-2]=\frac{1}{2}$ and $P[X+Y=2]=P[X+Y=-2]=\frac{1}{4}$, $P[X+Y=0]=\frac{1}{2}$).

b) Assume that X_n converges to X in distribution; then $E[f(X_n)]\to E[f(X)]$ for all bounded continuous functions f. Thus

$$\phi_{X_n}(t)=E\left[e^{itX_n}\right]=E[\cos(tX_n)+i\sin(tX_n)]\to E[\cos(tX)+i\sin(tX)]=\phi_X(t)$$

as $x\mapsto\sin(tx)$ and $x\mapsto\cos(tx)$ are bounded, continuous functions.

c) As X_n, Y_n and X, Y are mutually independent, we have

$$\phi_{X_n+Y_n}(t)=\phi_{X_n}(t)\phi_{Y_n}(t)\to\phi_X(t)\phi_Y(t)=\phi_{X+Y}(t)\ .$$

By Lévy's Continuity Theorem, X_n+Y_n converges in distribution to $X+Y$ (the condition that ϕ_{X+Y} is continuous at 0 is satisfied since ϕ_X and ϕ_Y are continuous at 0).

Problem 4: a) Let M be the maximum of $|a_1|,|a_2|,\ldots,|a_k|$. Then

$$|s_n-s_n^k|=\left|\frac{a_1+a_2+\cdots+a_k}{\sqrt{n}}\right|\le\frac{Mk}{\sqrt{n}}\ . \tag{1}$$

It suffices to show that for every $\epsilon>0$, we have

$$|\limsup_{n\to\infty}s_n-\limsup_{n\to\infty}s_n^k|\le\epsilon\ .$$

Given an ϵ, inequality (1) above shows us that there is an N such that $|s_n-s_n^k|<\epsilon$ when $n\ge N$. This means that $|\sup_{m\ge n}s_m-\sup_{m\ge n}s_m^k|\le\epsilon$ for all $n\ge N$. But then $|\lim_{n\to\infty}\sup_{m\ge n}s_m-\lim_{n\to\infty}\sup_{m\ge n}s_m^k|\le\epsilon$, which, by definition of lim sup, is just another way of saying that $|\limsup_{n\to\infty}s_n-\limsup_{n\to\infty}s_n^k|\le\epsilon$.

b) Since the random variables

$$S_n^k=\frac{X_k+X_{k+1}+\ldots+X_n}{\sqrt{n}}$$

are $\mathcal{F}_k^*=\sigma(X_k,X_{k+1},\ldots)$-measurable, so is $\limsup_{n\to\infty}S_n^k$. By a),

$$\limsup_{n\to\infty}S_n=\limsup_{n\to\infty}S_n^k\ ,$$

and hence $\limsup_{n\to\infty}S_n$ is $\mathcal{F}_k$-measurable for all k, which means that it is measurable with respect to the tail σ-algebra $\mathcal{F}_\infty^*$. Hence

$$\Lambda=\{\omega\,:\,\limsup_{n\to\infty}S_n(\omega)\in B\}$$

belongs to $\mathcal{F}^*_\infty$, and is a tail event. By Borel/Kolmogorov's Zero-One Law (Theorem 5.22), Λ can only have probability 0 or 1.