STKMAT3710: Trial Exam 3, Fall 2109

Problem 1: Let X be the binomial random varaible given by $P[X = -1] = P[X = 1] = \frac{1}{2}$.

a) Find a real expression for the characteristic function of X.

b) Assume that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of independent random variables with the same distribution as X. Find the characteristic function of

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \ .$$

c) Use the result in b) to show directly that S_n converges in distribution to a normal distribution ("show directly" means that you are not allowed to use a version of the Central Limit Theorem).

Problem 2: Let $\{X_1, X_2, \ldots, X_N\}$ be integrable, independent, and identically distributed random variables, and put $Y_0 = 0$ and $Y_n = \sum_{k=1}^{n} X_k$ for $1 \leq k \leq N$. Define $T\colon \Omega \to \{0, 1, 2, \ldots, N\}$ by

$$T(\omega) = \begin{cases} \text{the smallest n such that } Y_n \geq 10 \text{ if such an } n \text{ exists} \\ \\ N \text{ otherwise} \end{cases}$$

Let $m = E[X_i]$ be the common mean for the X_i's. Look at the three cases $m \geq 0$, $m = 0$, and $m \leq 0$. What can you say about the sign of $E[Y_T]$ in each case?

Problem 3: In this problem you may need the following version of the Fourier Inversion Formula: If f is a density function with an integrable characteristic function ϕ, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-ixt} \, dt$$

a) Let ϕ be the characteristic function of a random variable Y. Show that if $t\phi(t)$ is integrable, then Y has a differentiable density function ϕ.

b) Let X and Z_ϵ be two independent random variables, where Z_ϵ is normally distributed with mean 0 and variance ϵ^2. Let $Y_\epsilon = X + Z_\epsilon$. Show that $Y_\epsilon = X + Z_\epsilon$ has a differentiable density function.

c) Show that Y_ϵ converges in distribution to X as $\epsilon \to 0$.

Problem 4: In this problem $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of random variables converging in distribution to Y. Let F_Y and F_{X_n} be the distribution functions of Y and X_n, respectively.

a) Show that if a is a continuity point for the distribution function F_Y of Y, then $\lim_{n\to\infty} F_{X_n}(a-) = F_Y(a)$ (recall that $F(a-) = \lim_{x\uparrow a} F(x)$).

b) Show that

$$P[\limsup X_n \geq a] \geq P[Y \geq a]$$

for all $a \in \mathbb{R}$.

Problem 5.

a) Each day we are going to choose a number at random. The first day we choose the number from the set $\{1\}$, the second day from the set $\{1, 2\}$, the third day from the set $\{1, 2, 3\}$ etc. The choices are independent and each day all available numbers are equally probable. What is the probability that 1 gets chosen infinitely many times?

b) We repeat the experiment, but now we on day n choose from the set $\{1, 2, \ldots, n^2\}$ instead. What is now the probability of getting infinitely many 1's?