STKMAT3710: Trial Exam 3, Fall 2109. Solution

Problem 1: a) The characteristic function is

$$\phi_X(t) = E[e^{itX}] = e^{it\cdot 1}\frac{1}{2} + e^{it\cdot(-1)}\frac{1}{2} = \cos t.$$

b)Using the independence, we have

$$\phi_{S_n}(t) = E\left[e^{itS_n}\right] = E\left[e^{i\frac{t}{\sqrt{n}}X_1}e^{i\frac{t}{\sqrt{n}}X_2}\cdot\ldots\cdot e^{i\frac{t}{\sqrt{n}}X_n}\right]$$

$$= E\left[e^{i\frac{t}{\sqrt{n}}X_1}\right]E\left[e^{i\frac{t}{\sqrt{n}}X_2}\right]\cdot\ldots\cdot E\left[e^{i\frac{t}{\sqrt{n}}X_n}\right] = \left(\phi(\frac{t}{\sqrt{n}})\right)^n = \left(\cos(\frac{t}{\sqrt{n}})\right)^n .$$

c) The Taylor expansion for cosine is $\cos x = 1 - \frac{x^2}{2} + o(x^2)$, and hence

$$\phi_{S_n}(t) = \left(\cos(\frac{t}{\sqrt{n}})\right)^n = \left(1 - \frac{t^2}{2n} + o(\frac{t^2}{n})\right)^n = \left(1 - \frac{t^2}{2n} + o(\frac{t^2}{n})\right)^n .$$

Consequently (e.g. by Lemma 6.34)

$$\phi_{S_n}(t) = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n \to e^{-\frac{t^2}{2}}.$$

As $e^{-\frac{t^2}{2}}$ is the characteristic function of a normal distribution with mean 0 and variance $\sigma^2 = 1$, the result follows from Lévy's Continuity Theorem.

Problem 2: Put $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma\{X_1, X_2, \ldots, X_k\}$ for $k \in \{1, 2, \ldots, N\}$. Then

$$E[Y_{k+1}|\mathcal{F}_k] = E[Y_k + X_{k+1}|\mathcal{F}_k] = E[Y_k|\mathcal{F}_k] + E[X_{k+1}|\mathcal{F}_k] = Y_k + m$$

as Y_k is $\mathcal{F}_k$-measurable and X_{k+1} is independent of $\mathcal{F}_k$ (and hence $E[X_{k+1}|\mathcal{F}_k] = E[X_{k+1}] = m$). This shows that $\{Y_n\}$ is a submartingale if $m \geq 0$, a martingale if $m = 0$, and a supermartingale if $m \leq 0$.

Note that T is a bounded $\{\mathcal{F}_n\}$-stopping time and so (obviously) is the constant time 0. According to Theorem 9.9, (Y_0, Y_T) is a $(\mathcal{F}_0, \mathcal{F}_T)$-submartingale/martingale/supermartingale according to whether Y is a submartingale/martingale/supermartingale. Hence (Y_0, Y_T) is a submartingale if $m \geq 0$, a martingale if $m = 0$, and a supermartingale if $m \leq 0$. As $E[Y_T|\mathcal{F}_0] = E[Y_T]$ since $\mathcal{F}_0$ is trivial, we get:

(i) $E[Y_T] = E[Y_T|\mathcal{F}_0] \geq E[Y_0] = 0$ if $m \geq 0$.

(ii) $E[Y_T] = E[Y_T|\mathcal{F}_0] = E[Y_0] = 0$ if $m = 0$.

(iii) $E[Y_T] = E[Y_T|\mathcal{F}_0] \leq E[Y_0] = 0$ if $m \leq 0$.

Problem 3: a) As $t\phi(t)$ is integrable, so is $\phi(t)$, and we can use the formula in the problem to compute the derivative of f:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \int_{-\infty}^{\infty} \phi(t) \frac{e^{-i(x+h)t} - e^{-ixt}}{h} dt$$

$$= \lim_{h\to 0} \int_{-\infty}^{\infty} \phi(t) e^{ixt} \frac{e^{-iht}-1}{h} dt = \lim_{h\to 0} \int_{-\infty}^{\infty} t\phi(t) e^{-ixt} \frac{e^{-iht}-1}{ht} dt$$

Let us take a closer look at the only part of the integrand that depends on h:

$$\frac{e^{-iht}-1}{ht} = \frac{\cos(-ht)-1}{ht} + i\frac{\sin(-ht)}{ht} = \frac{\cos(ht)-1}{ht} - i\frac{\sin(ht)}{ht}$$

By the Mean Value Theorem, there are numbers c_1 and c_2 between 0 and ht such that

$$\frac{e^{-iht}-1}{ht} = \frac{\cos(ht)-1}{ht} - i\frac{\sin(ht)}{ht} = -\sin(c_1) - i\cos(c_2)$$

Hence

$$f'(x) = \lim_{h\to 0} \int_{-\infty}^{\infty} t\phi(t)e^{-ixt}(-\sin(c_1) - i\cos(c_2))\, dt$$

As the integrand converges to $-it\phi(t)e^{ixt}$ and is bounded by the integrable function $|t\phi(t)|$, the Dominated Convergence Theorem tells us that

$$f'(x) = \lim_{h\to 0} \int_{-\infty}^{\infty} t\phi(t)e^{-ixt}(-\sin(c_1) - i\cos(c_2))\, dt$$

$$= \int_{-\infty}^{\infty} \lim_{h\to 0} \left(t\phi(t)e^{-ixt}(-\sin(c_1) - i\cos(c_2))\right)\, dt = \int_{-\infty}^{\infty} -it\phi(t)e^{-ixt}\, dt$$

Hence f is differentiable.

b) The characteristic function of Z_ϵ is $\phi_{Z_\epsilon}(t) = e^{-\frac{\epsilon^2 t^2}{2}}$. By independence,

$$\phi_{Y_\epsilon}(t) = \phi_X(t)\phi_{Z_\epsilon}(t) = \phi_X(t)e^{-\frac{\epsilon^2 t^2}{2}}.$$

As

$$|t\phi_{Y_\epsilon}(t)| = |t\phi_X(t)e^{-\frac{\epsilon^2 t^2}{2}}| \le \left(|t|e^{-\frac{\epsilon^2 t^2}{4}}\right) e^{-\frac{\epsilon^2 t^2}{4}} \le e^{-\frac{\epsilon^2 t^2}{4}}$$

for large $|t|$, and $e^{-\frac{\epsilon^2 t^2}{4}}$ is integrable, we see that $t\phi_{Y_\epsilon}(t)$ is integrable, and hence by a), Y_ϵ has a differentiable density function.

c) As $\lim_{\epsilon\to 0}\phi_{Y_\epsilon}(t) = \lim_{\epsilon\to 0}\phi_X(t)e^{-\frac{\epsilon^2 t^2}{2}} = \phi_X(t)$, Lévy's Continuity Theorem tells us that Y_ϵ converges in distribution to X.

Problem 4: a) Let $x < a$ be a continuity point of F_Y. Then $F_{X_n}(x) \le F_{X_n}(a-) \le F_{X_n}(a)$, and we have

$$F_Y(x) = \lim_{n\to\infty} F_{X_n}(x) \le \liminf F_{X_n}(a-) \le \limsup F_{X_n}(a-) \le \lim_{n\to\infty} F_{X_n}(a) = F_Y(a)$$

As a is a continuity point of F_Y, we can get $F_Y(x)$ as close to $F_Y(a)$ as we wish, and hence $\liminf F_{X_n}(a-) = \limsup F_{X_n}(a-) = F_Y(a)$.

b) If a is a continuity point for F_Y, we have

$$P[\limsup X_n \geq a] = P[\lim_{n\to\infty} \sup_{k\geq n} X_k \geq a]$$

$$= P[\bigcap_{n=1}^{\infty} \{\omega : \sup_{k\geq n} X_k \geq a\} = \lim_{n\to\infty} P[\sup_{k\geq n} X_k \geq a] \geq \lim_{n\to\infty} P[X_n \geq a]$$

$$= \lim_{n\to\infty} (1 - F_{X_n}(a-)) = 1 - F_Y(a) = P[Y \geq a]$$

where we used part a) in the next to last step. As the continuity points are dense, and distribution functions are right continuous, we have $P[\limsup X_n \geq a] \geq P[Y \geq a]$ for all a.

Problem 5. a) Let B_n be the event "number 1 is chosen on day n". Then the B_n's are independent and $P(B_n) = \frac{1}{n}$. As $\sum_{n=1}^{\infty} P(B_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, the converse Borel-Cantelli Lemma tells us that $P(\limsup B_n) = 1$, and hence 1 is chosen infinitely many times with probability 1.

b) Let C_n be the event "number 1 is chosen on day n". Then the C_n's are independent and $P(C_n) = \frac{1}{n^2}$. As $\sum_{n=1}^{\infty} P(C_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, the Borel-Cantelli Lemma tells us that $P(\limsup C_n) = 0$, and hence 1 is chosen infinitely many times with probability 0.