30th September, 2019

STK-MAT3710/4710

Mandatory assignment 1 of 1

Submission deadline

Thursday 17th October 2019, 14:30 in Canvas (canvas.uio.no).

Instructions

If you are taking STK-MAT3710, you can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). If you are taking STK-MAT4710, you have to write the assignment in LaTeX. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. You need a 60 % score to pass this assignment. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

Problem 1. Assume that $X_1, X_2, \ldots, X_n$ are independent random variables with finite 4^{th} moments and with expectation 0 (i.e. $E(X_i) = 0$ for all i). Show that

$$E\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = \sum_{r=1}^{n} E\left[X_r^4\right] + 6\sum_{q=1}^{n}\sum_{p=1}^{q-1} E\left[X_p^2\right] E\left[X_q^2\right]$$

Problem 2. In this problem Ω is a non-empty set, and all complements are with respect to Ω, i.e. $A^c = \Omega \setminus A$. We look at a family $\mathcal{D}$ of subsets of Ω satisfying the following conditions:

(i) $\emptyset \in \mathcal{D}$

(ii) If $A \in \mathcal{D}$, then $A^c \in \mathcal{D}$.

(iii) If $\{B_n\}$ is a pairwise disjoint sequence of sets in $\mathcal{D}$ (i.e. $B_i \cap B_j = \emptyset$ for $i \neq j$), then $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{D}$.

Such a family $\mathcal{D}$ is called a *D-system*.

a) Show that for all sets $A, B \subseteq \Omega$, we have $A \setminus B = (A^c \cup B)^c$.

b) Show that if $A, B \in \mathcal{D}$ and $B \subseteq A$, then $A \setminus B \in \mathcal{D}$.

c) Show that if $\{A_n\}$ is an increasing sequence of sets in $\mathcal{D}$ (i.e. $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$), then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{D}$.

d) Give an example of a D-system that is not a σ-algebra.

e) Show that if D is a D-system that is closed under finite intersections (i.e. if $A, B \in \mathcal{D}$, then $A \cap B \in \mathcal{D}$), then $\mathcal{D}$ is a σ-algebra.

Problem 3. In this problem we are going to study infinite sequences of coin tosses. We let Ω be the set of all sequences $\omega = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}$, where each ω_n is either H (for *heads*) or T (for *tails*). If $I = \{i_1, i_2, \ldots, i_n\}$ is a set of n different natural numbers in increasing order, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a n-tuple of H's and T's, we let

$$C_{I,\alpha} = \{\omega \in \Omega \,:\, \omega_{i_1} = \alpha_1, \omega_{i_2} = \alpha_2, \ldots, \text{ and } \omega_{i_n} = \alpha_n\}$$

A set of this form is called a *cylinder set of length n*.

Our probability space is going to be $(\Omega, \mathcal{C}, P)$, where $\mathcal{C}$ is the σ-algebra generated by *all* cylinder sets (regardless of length), and P is the probability measure on $\mathcal{C}$ such that if $C_{I,\alpha}$ is a cylinder set of length n, then $P(C_{I,\alpha}) = \frac{1}{2^n}$ (there is a unique such measure).

a) Show that two cylinder sets $C_{I,\alpha}$ and $C_{J,\beta}$ are independent if and only if $I \cap J = \emptyset$.

b) Show that for almost all ω, the sequence $\omega = \{\omega_1, \omega_2, \dots, \omega_n, \dots\}$ contains an infinite number of H's.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be an n-tuple of H's and T's. We say that α *occurs* in $\omega = \{\omega_1, \omega_2, \dots, \omega_n, \dots\}$ if there is an i such that $\omega_{i+1} = \alpha_1$, $\omega_{i+2} = \alpha_2$, $\dots$, $\omega_{i+n} = \alpha_n$.

c) Fix an n-tuple α. Show that for almost all ω, the tuple α occurs infinitely many times in ω.

d) Show that there is a set $\Omega' \subseteq \Omega$ with $P(\Omega') = 1$ such that if $\omega \in \Omega'$, then *all* n-tuples α (regardless of their length n) occur infinitely many times in ω.

Good Luck!