σ-algebras generated by functions

Recall: If $X : \Omega \to \mathbb{R}$, then $\sigma(X)$ is the smallest σ-algebra containing $\{\omega : X(\omega) \le x\}$ for all $x \in \mathbb{R}$.
Observe that X is $\sigma(X)$-measurable.

Theorem: A set $A \subseteq \Omega$ is in $\sigma(X)$ if and only if $A = X^{-1}(B)$ for a Borel set B. In other words

$$\sigma(X) = X^{-1}(\mathcal{B})$$

where $\mathcal{B}$ is the family of Borel sets.

Proof: Let $\mathcal{H} = X^{-1}(\mathcal{B}) = \{X^{-1}(B) : B \in \mathcal{B}\}$, then $\mathcal{H}$ is a σ-algebra as it is the inverse image of the σ-algebra $\mathcal{B}$. Since

$$X^{-1}((-\infty, x]) \in \mathcal{H},$$

we must $\underline{\sigma(X) \subseteq \mathcal{H}}$ as $\sigma(X)$ is the smallest σ-algebra containing $X^{-1}((-\infty,x]) = \{\omega : X(\omega) \le x\}$.

On the other hand, since X is $\sigma(X)$-measurable, we know that $\underbrace{X^{-1}(B)}_{\mathcal{H}} \in \sigma(X)$ for any Borel set B.

Thus $\mathcal{H} \subseteq \sigma(X)$, and it follows that $\sigma(X) = \mathcal{H}$.

Theorem: Let $X, Y : \Omega \to \mathbb{R}$. Then Y is $\sigma(X)$-measurable if and only if there is a Borel function $f : \mathbb{R} \to \mathbb{R}$ such $Y = f(X)$.

Proof: Assume that $Y = f(X)$ for a Borel function f.

$$\{\omega : Y(\omega) \le x\} = \{\omega : f(X(\omega)) \le x\}$$

$$= \{\omega : X(\omega) \in \underbrace{f^{-1}((-\infty, x])}_{\text{Borel-set}}\} \in \sigma(X)$$

since X is $\sigma(X)$-measurable

Assume now that Y is $\sigma(X)$-measurable. We need to find a Borel function f such that $Y = f(X)$. Let

$$Y_n(\omega) = \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \underbrace{\mathbb{1}_{\left(\frac{k}{2^n} < Y(\omega) \le \frac{k+1}{2^n}\right)}}_{\text{is in } \sigma(X) \text{ as } Y \text{ is } \sigma(X)\text{-measurable.}}(\omega)$$

From the previous theorem we know that

$$\{\omega : \tfrac{k}{2^n} < Y(\omega) \le \tfrac{k+1}{2^n}\} = X^{-1}(B_{n,k})$$

for a Borel set $B_{n,k}$. Define f_n by

$$f_n(x) = \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \mathbb{1}_{B_{n,k}} \quad \text{Borel-function}$$

Observe that

$$f_n(X(\omega)) = \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \mathbb{1}_{\{\omega : X(\omega) \in B_{n,k}\}}$$

$$= \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \mathbb{1}_{\left(\frac{k}{2^n} < Y(\omega) \le \frac{k+1}{2^n}\right]} = Y_n(\omega)$$

Note that $f_n(X(\omega)) \uparrow Y(\omega)$. Also note that

$f_n(x) \le f_{n+1}(x) \le f_n(x) + \frac{1}{2^n}$ (requires some care in the choice of $B_{n,k}$)

Hence $f_n(x) \uparrow f(x)$, where f is a Borel function.

This means that $f(X) = \lim f_n(X) = Y$.

Independence of σ-algebras

<u>Definition</u>: Two σ-algebras $\mathcal{G}$ and $\mathcal{F}$ on Ω are <u>independent</u> if

$$P(G\cap F)=P(G)P(F)$$

for all $G\in\mathcal{G}$ and all $F\in\mathcal{F}$.

<u>Definition</u>: A finite family $\mathcal{F}_1,\mathcal{F}_2,\dots,\mathcal{F}_n$ of σ-algebras is <u>independent</u> if whenever $F_1\in\mathcal{F}_1$, $F_2\in\mathcal{F}_2,\dots, F_n\in\mathcal{F}_n$, then

$$P(F_1\cap F_2\cap\dots\cap F_n)=P(F_1)P(F_2)\dots P(F_n).$$

<u>Definition</u>: An infinite family $\{\mathcal{F}_i\}_{i\in I}$ of σ-algebras is <u>independent</u> if all finite subfamilies are independent.

<u>Proposition</u>: Two random variables X and Y are independent if and only if $\sigma(X)$ and $\sigma(Y)$ are independent.

<u>Proof</u>: Assume that $\sigma(X)$ and $\sigma(Y)$ are independent. For all $x,y\in\mathbb{R}$, the sets $\{X\le x\}$ and $\{Y\le y\}$ are $\sigma(X)$ and $\sigma(Y)$, respectively. Hence

$$P[\underbrace{\{X\le x\}}_{\sigma(X)}\cap\underbrace{\{Y\le y\}}_{\sigma(Y)}]=P[X\le x]P[Y\le y]$$

hence X,Y are independent.

Assume next that X and Y are independent, and let $B_1\in\sigma(X)$ and $B_2\in\sigma(Y)$. We must show that

$$P[B_1\cap B_2]=P[B_1]P(B_2).$$

Since $B_1\in\sigma(X)$, ~~but~~ we have $B_1=X^{-1}(A_1)$ for a Borel set A_1; i.e. $B_1=\{\omega: X(\omega)\in A_1\}$

Correspondingly, since $B_2\in\sigma(Y)$, there is a Borel set A_2 s.t. $B_2=\{\omega: Y(\omega)\in A_2\}$.

$$P[B_1\cap B_2]=P[\{X\in A_1\}\cap\{Y\in A_2\}]$$

$$\overset{\text{independence of }X,Y}{=}P\{\underbrace{X\in A_1}_{B_1}\}P\{\underbrace{Y\in A_2}_{B_2}\}=P(B_1)P(B_2)$$

Problem on Exam 2019

1 Throw a die infinitely many times

Consecutive 6's $\ldots\ldots 6\ \underbrace{6\ 6 \ldots\ldots\ldots 6}_{17 \text{ times}}$

What is the prob. of having infinitely many occurrences of 17 consecutive 6's.

$\underbrace{6\ 6 \ldots\ldots 6}_{A_1}\ \underbrace{6\ 6 \ldots\ldots\ldots 6}_{A_2}\ \underbrace{\qquad\qquad}$

A_n = the event of getting 17 consecutive 6's in period n.

$$P(A_n) = \left(\frac{1}{6}\right)^{17} \qquad \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{17} = \infty$$

B.C. $P[\limsup A_n] = 1$

The probability of getting 17 consecutive 6's infinitely many of in the periods is one. Hence the prob of infinitely many series of 6 regardless of the fixed periods must also be 1.

Problem 5 · Trial Exam 3

Every day we pick a number at random. What is the probability of getting infinitely ones?

a) Day 1 $\{1\}$

Day 2 $\{1,2\}$

⋮

Day n $\{1,2,\dots,n\}$

$P(A_n) = P(\text{getting 1 on day } n)$

$= \frac{1}{n}$

$\sum P(A_n) = \sum \frac{1}{n} = \infty.$

BC: $P[\limsup A_n] = 1$

Prob. of getting infinitely many 1's is 1

b) Day 1 $\{1\}$

Day 2 $\{1,2,3,4\}$

⋮

Day n $\{1,2,\dots,n^2\}$

$P(A_n) = P(\text{getting 1 on day } n)$

$= \frac{1}{n^2}$

$\sum P(A_n) = \sum \frac{1}{n^2} < \infty$

BC:

$P[\limsup A_n] = 0$

Prob of getting inf. many 1's is 0.

Ex 4.3: X integrable $P(\Lambda_n) \to 0$

Show $\int_{\Lambda_n} X \, dP \to 0$.

Let $X_n = \mathbb{1}_{\Lambda_n} X$, bounded by $|X|$ integrable.

Observe $X_n \to 0$ in prob., $|X_n| \leq |X|$

Dom. Conv. Th. for convergence prob:

$$\left.\begin{aligned} &\lim_{n\to\infty} E[X_n] = E[0] = 0. \\ &E[X_n] = \int \mathbb{1}_{\Lambda_n} X \, dP = \int_{\Lambda_n} X \, dP \end{aligned}\right\} \int_{\Lambda_n} X \, dP \to 0$$

Ex 4.7:

$$\left.\begin{aligned} &X_n \to X \quad \text{prob} \\ &X_n \to Y \quad \text{prob} \end{aligned}\right\} \Rightarrow X = Y \quad \text{a.s.}$$

$$\varepsilon \leq |X - Y| \leq \underbrace{|X - X_n| + |X_n - Y|}$$

at least one is larger than $\varepsilon/2$.

$$P[|X-Y| \geq \varepsilon] \leq \underbrace{P[|X - X_n| \geq \tfrac{\varepsilon}{2}]}_{\to 0} + \underbrace{P[|X_n - Y| \geq \tfrac{\varepsilon}{2}]}_{\to 0}$$

hence $P[|X-Y| \geq \varepsilon] = 0$ for all ε.

This implies that $X = Y$ a.e.

To see this, put $\varepsilon = \frac{1}{m}$

$$\underbrace{\{\omega : X \neq Y\}}_{P=0} = \underbrace{\bigcup_{m\in\mathbb{N}} \{|X-Y| \geq \tfrac{1}{m}\}}_{P=0}$$