Types of distributions

Definition: A random variable X has a discrete distribution if there is a countable set $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\sum_{n\in\mathbb{N}} P(\{x_n\}) = 1 \quad \left(\text{hence} \quad P\Big(\Omega \setminus \bigcup_{n\in\mathbb{N}}\{x_n\}\Big) = 0\right)$$

Example: Let $\{q_n\}$ be an enumeration of the rational numbers. If X is a random variable which takes the values q_n with prob.

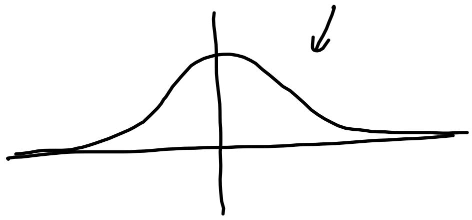
$$P\{\omega : X(\omega) = q_n\} = \frac{1}{2^n}$$

Definition: A r.v. X has continuous distribution if the distribution function is continuous

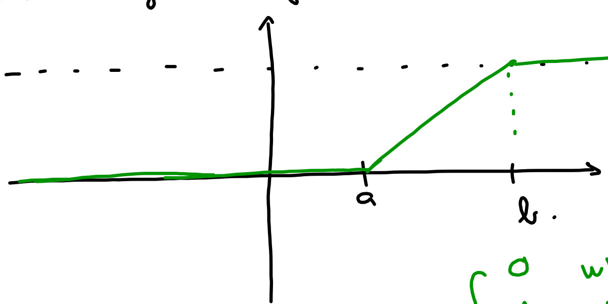
Definition: X has absolutely continuous distribution if there is a (density) function f such that

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Example: Gaussian distribution $f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$



Example: The uniform distribution $U[a,b]$ on the interval $[a,b]$ is given by



$$F(x) = \begin{cases} 0 & \text{when } x \le a \\ 1 & \text{when } x \ge b \\ \frac{x-a}{b-a} & \end{cases}$$

Expectations

Aim: To define the mean or average value of a random variable X : $E[X]$.

Strategy: Find discrete X $\overset{\text{approx}}{\Longrightarrow}$ generalize to general X

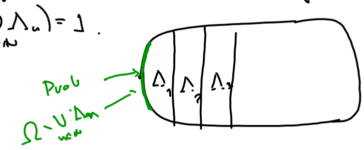
Recall from MAT1110: $\sum_{n=1}^{\infty} a_n$

(i) The series $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{N\to\infty}\sum_{n=1}^{N} a_n$ exists, and we then define the sum as $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty}\sum_{n=1}^{N} a_n$

(ii) $\sum_{n=1}^{\infty} a_n$ converge absolutely if $\sum_{n=1}^{\infty}|a_n|$ converges (this implies that $\sum_{n=1}^{\infty} a_n$ converges). If $\sum a_n$ is convergent, but not absolutely, then it is conditionally convergent.

Assume that X is a discrete random variable and let $\{x_n\}$ be the (distinct) points such $P\{X = x_n\} > 0$.
Then $\sum_{n=1}^{\infty} P\{x_n\} = 1$.

Definition A quasi-partition of Ω is a collection $\{\Lambda_n\}_{n\in\mathbb{N}}$ of events such that $\Lambda_i \cap \Lambda_j = \emptyset$ when $i \neq j$ and $P(\bigcup_{n\in\mathbb{N}} \Lambda_n) = 1$.



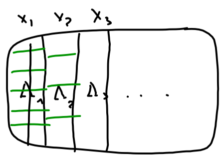
Typically $\Lambda_n = \{\omega : X(\omega) = x_n\}$

Definition: Let X be a discrete random variable taking the values $\{x_n\}$ with pos. prob. Let
$$\Lambda_n = \{\omega : X(\omega) = x_n\}$$
We say that X is integrable if
$$\sum_{n=1}^{\infty} |x_n| P(\Lambda_n) < \infty,$$
and if so we define the expectation of X by
$$E[X] = \sum_{n=1}^{\infty} x_n P(\Lambda_n).$$

Intuitively $E[X] = \sum_{n\in\mathbb{N}} x_n P(\Lambda_n)$

Proposition: Let $\{\Gamma_i\}$ be a quasi-partition of Ω such that X is constant on each Γ_i with value x_i. Then X is integrable if and only if
$$\sum |x_i| P(\Gamma_i) < \infty$$
and then
$$E[X] = \sum_i x_i P(\Gamma_i)$$

Sketch of proof: Since X is constant on the Γ_i's, each Γ_i lies inside a Λ_n.



$$\sum |x_i| P(\Gamma_i) = \sum_{n=1}^{\infty} \sum_{\Gamma_i \subseteq \Lambda_n} |x_i| P(\Gamma_i)$$
$$= \sum_{n=1}^{\infty} |x_n| \overbrace{\sum_{\Gamma_i \subseteq \Lambda_n} P(\Gamma_i)}^{P(\Lambda_n)} = \sum_{n=1}^{\infty} |x_n| P(\Lambda_n)$$

Hence X is integrable (i.e. $\sum_{n=1}^{\infty} |x_n| P(\Lambda_n) < \infty$) iff $\sum |x_i| P(\Gamma_i) < \infty$.

For the second, repeat the first part without absolute values.
$$\sum_i x_i P(\Gamma_i) = \sum_n \sum_{\Gamma_i \subseteq \Lambda_n} \overset{x_n}{x_i} P(\Gamma_i) = \sum x_n \sum_{\Gamma_i \subseteq \Lambda_n} P(\Gamma_i)$$
$$= \sum x_n P(\Lambda_n) = E[X].$$

Problems

Extra problem 1: TFAE:

(i) X is a random variable ($\{\omega : X(\omega) \le x\} \in \mathcal{F}$ for all x)

(ii) $\{\omega : X(\omega) < x\} \in \mathcal{F}$ for all x

(iii) $\{\omega : X(\omega) > x\} \in \mathcal{F}$ for all x

(iv) $\{\omega : X(\omega) \ge x\} \in \mathcal{F}$ for all x.

Proof: (i) $\Rightarrow$ (ii)

$$\{\omega : X(\omega) < x\} = \bigcup_{n \in \mathbb{N}} \overbrace{\{\omega : X(\omega) \le x - \tfrac{1}{n}\}}^{\mathcal{F}} \in \mathcal{F}.$$

(ii) $\Rightarrow$ (i)

$$\{\omega : X(\omega) \le x\} = \bigcap_{n \in \mathbb{N}} \overbrace{\{\omega : X(\omega) < x + \tfrac{1}{n}\}}^{\mathcal{F}} \in \mathcal{F}.$$

(i) $\Rightarrow$ (iii)

$$\{\omega : X(\omega) > x\} = \underbrace{\{\omega : X(\omega) \le x\}}_{\mathcal{F}}{}^{c} \in \mathcal{F}.$$

(iii) $\Rightarrow$ (i) $\{\omega : X(\omega) \le x\} = \underbrace{\{\omega : X(\omega) > x\}}_{\mathcal{F}}{}^{c} \in \mathcal{F}.$

(ii) $\Leftrightarrow$ (iv) use complements in the same way.

Extra problem 2: X is r.v. Define

$$Y(\omega) = \begin{cases} \frac{1}{X(\omega)} & \text{if } X(\omega) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Prove that Y is a r.v.

Want to prove that $\{\omega : Y(\omega) > x\} \in \mathcal{F}$ for all x.

(i) $x > 0$: $\{\omega : Y(\omega) > x\} = \{\omega : \frac{1}{X(\omega)} > x\} = \{\omega : 0 < X(\omega) < \frac{1}{x}\}$
$\in \mathcal{F}$.

(ii) $x = 0$: $\{\omega : Y(\omega) > 0\} = \{\omega : X(\omega) > 0\} \in \mathcal{F}$.

(iii) $x < 0$: $\{\omega : \underbrace{Y(\omega) > x}_{\frac{1}{X(\omega)}}\} = \{\omega : X(\omega) \geq 0\} \cup$

$\{\omega : X(\omega) < 0 \text{ and } X(\omega) < \frac{1}{x}\}$

$$Z(\omega) = \begin{cases} \frac{Y(\omega)}{X(\omega)} & \text{when } X(\omega) \neq 0 \\ 0 & \text{when } X(\omega) = 0. \end{cases}$$

$Y(\omega) \cdot \frac{1}{X(\omega)}$

Ex 2.2, page 42: A function $f:\mathbb{R}\to\mathbb{R}$ is Borel measurable / a Borel function if

$$f^{-1}(-\infty,\alpha] = \{x : f(x)\le \alpha\} \in \mathcal{B} \text{ for all } \alpha\in\mathbb{R}$$

↑ Borel σ-algebra.

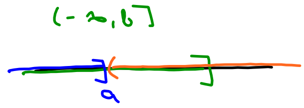
(i) Prove that if f is a Borel function then $f^{-1}(B)\in\mathcal{B}$ for all $B\in\mathcal{B}$.

Let

$$\mathcal{A} = \{B\in\mathcal{B} : f^{-1}(B)\in\mathcal{B}\}$$

Plan: Prove that $\mathcal{A}$ is a σ-algebra containing all half-open intervals $(a,b]$. Since $\mathcal{B}$ is the smallest such σ-algebra, we then have $\mathcal{B}\subseteq\mathcal{A}\subseteq\mathcal{B}$.

$\mathcal{A}$ contains all half-open intervals:



$$f^{-1}(a,b] = f^{-1}\big((-\infty,b]\cap(-\infty,a]^c\big)$$
$$= \underbrace{f^{-1}((-\infty,b])}_{\mathcal{B}} \cap \underbrace{\big(\underbrace{f^{-1}(-\infty,a]}_{\mathcal{B}}\big)^c}_{\mathcal{B}} \in \mathcal{B}.$$

Check that $\mathcal{A}$ is a σ-algebra:

(i) $\emptyset\in\mathcal{A}$ because: $f^{-1}(\emptyset)=\emptyset\in\mathcal{B}$

(ii) $B\in\mathcal{A}$, need to prove $B^c\in\mathcal{A}$.

$$f^{-1}(B^c) = (\underbrace{f^{-1}(B)}_{\mathcal{B}})^c \in\mathcal{B} \text{ and hence } B^c\in\mathcal{A}.$$

(iii) Assume $A_n\in\mathcal{A}$ for all n, need to prove that $\bigcup A_n\in\mathcal{A}$.

$$f^{-1}\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \bigcup_{n\in\mathbb{N}} \underbrace{f^{-1}(A_n)}_{\mathcal{B}} \in\mathcal{B}, \text{ and hence } \bigcup A_n\in\mathcal{A}.$$

$$\mathcal{B} = \mathcal{A} = \{B\in\mathcal{B} : f^{-1}(B)\in\mathcal{B}\}$$

Hence the inverse image of any Borel set is a Borel set.

(ii) Show that if X is a r.v. and $f:\mathbb{R}\to\mathbb{R}$ is a Borel function, then $Y(\omega)=f(X(\omega))$ is a r.v.

We have

$$\{\omega : Y(\omega)\le x\} = \{\omega : f(X(\omega))\le x\} = \{\omega : f(X(\omega))\in(-\infty,x]\}$$
$$= \{\omega : X(\omega)\in \underbrace{f^{-1}(-\infty,x]}_{\text{Borel set.}}\}\in\mathcal{F}\ldots$$

by Prop 2.8.