Jensen's inequality for conditional expectations

Tools: $X_n \downarrow X, X_n \uparrow X \Rightarrow E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$ a.s.

Jensen inequality: $\sum_{n=1}^{\infty} p_n = 1$, $p_n \geq 0$. φ convex.

$$\varphi\Big(\sum_{n=1}^{\infty} x_n p_n\Big) \leq \sum_{n=1}^{\infty} \varphi(x_n) p_n.$$

Jensen's Inequality for conditional expectations: Assume $(\Omega, \mathcal{F}, P)$ is a probability space and that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. If $\varphi : (x_1, x_2) \to \mathbb{R}$ is convex and X is a random variable taking values in this interval s.t. X and $\varphi(X)$ are integrable, then

$$\varphi(E[X|\mathcal{G}]) \leq E[\varphi(X)|\mathcal{G}] \quad \text{a.s.}$$

Proof: Assume first that $X = \sum_{n=1}^{\infty} a_n 1\!\!1_{A_n}$ where $\{A_n\}$ is a measurable partition of Ω. Then

$$\varphi(E[X|\mathcal{G}]) = \varphi\Big(E\Big[\sum_{n=1}^{\infty} a_n 1\!\!1_{A_n}\Big|\mathcal{G}\Big]\Big) = \varphi\Big(\sum_{n=1}^{\infty} a_n \overbrace{E[1\!\!1_{A_n}|\mathcal{G}]}^{p_n(\omega)}\Big)$$

$$\leq \sum_{n=1}^{\infty} \varphi(a_n) E[1\!\!1_{A_n}|\mathcal{G}] = E[\varphi(X)|\mathcal{G}].$$

$$\boxed{\sum_{n=1}^{\infty} \underbrace{\overbrace{E[1\!\!1_{A_n}|\mathcal{G}]}^{p_n(\omega)}}_{\text{po. less than } 1}(\omega) = E\Big[\underbrace{\sum_{n=1}^{\infty} 1\!\!1_{A_n}}_{1}\Big|\mathcal{G}\Big] = 1}$$

If X is a general, integrable random variable, let X_n be the lower approximations to X. Then (by what we just proved)

$$(*) \quad \varphi(E[X_n|\mathcal{G}]) \leq E[\varphi(X_n)|\mathcal{G}]$$

$\downarrow$ $X_n \uparrow X$, $E[X|\mathcal{G}]$; φ is cont. $\quad \downarrow$? ok for increasing and decreasing.

$X_n \uparrow X$, $X_n \downarrow X$: $E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$

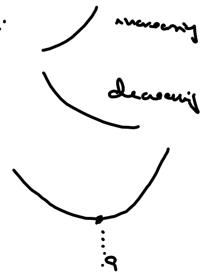
$$(**) \quad \varphi(E[X]|\mathcal{G}] \leq E[\varphi(X)|\mathcal{G}]$$

It remains to prove that

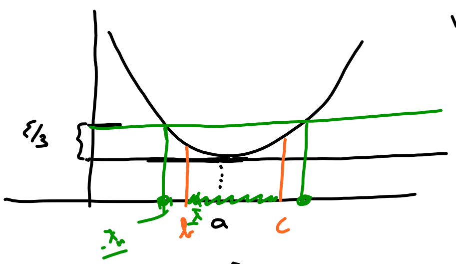
$$E[\varphi(X_n)|\mathcal{G}] \to E[\varphi(X)|\mathcal{G}]$$

in this case.

Convex: increasing, decreasing



Assume that $\varepsilon > 0$ is given. Since φ is cont, we can choose b and c as on the figure.



$$E[\varphi(X_n)|\mathcal{G}] = E[1\!\!1_{[X<b]}\varphi(X_n)|\mathcal{G}] + E[1\!\!1_{[b \leq X \leq c]}\varphi(X_n)|\mathcal{G}] + E[1\!\!1_{[X>c]}\varphi(X_n)|\mathcal{G}]$$

$$\downarrow\Big\} < \tfrac{\varepsilon}{3} \qquad \updownarrow \quad \Big| < \tfrac{\varepsilon}{3} \qquad \downarrow \ \varepsilon/3$$

$$E[\varphi(X)|\mathcal{G}] = E[1\!\!1_{[X<b]}\varphi(X)|\mathcal{G}] + E[1\!\!1_{[b \leq X \leq c]}\varphi(X)|\mathcal{G}] + E[1\!\!1_{[X>c]}\varphi(X)|\mathcal{G}]$$

Therefore for sufficiently large n, $|E[\varphi(X_n)|\mathcal{G}] - E[\varphi(X)|\mathcal{G}]| < \varepsilon$ and hence $E[\varphi(X_n)|\mathcal{G}] \to E[\varphi(X)|\mathcal{G}]$. The Theorem follows.

Ch 9: Martingales

Timeline $\mathbb{T}$: A filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$, increasing and $\mathcal{F}_t \subseteq \mathcal{F}$.

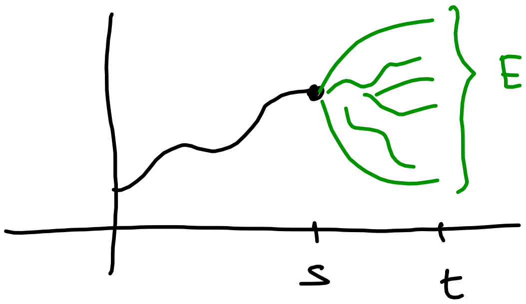
A process $X = \{X_t\}_{t\in\mathbb{T}}$ is adapted to $\{\mathcal{F}_t\}$ is X_t is $\mathcal{F}_t$-measurable for each t.

Define: Assume that X is an integrable and adapted process. Then

(i) X is a martingale if whenever $t > s$, then $E[X_t \mid \mathcal{F}_s] = X_s$ a.e.

(ii) —"— submartingale ——"—— $E[X_t \mid \mathcal{F}_s] \geq X_s$ a.s

(iii) —"— supermartingale ——"—— $E[X_t \mid \mathcal{F}_s] \leq X_s$ a.s.



$E[X_t \mid \mathcal{F}_s]$ average gain at time t.

$E[X_t \mid \mathcal{F}_s] = X_s$ mart.

$E[X_t \mid \mathcal{F}_s] \geq X_s$ submar.

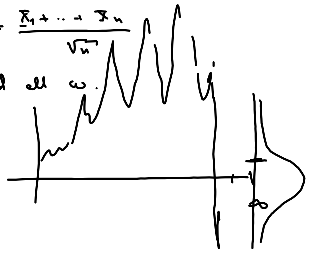
$E[X_t \mid \mathcal{F}_s] \leq X_s$ super.

Problems

p. 188, 6.29
p. 211, 7.16
Trial Exam 1, Problem 1

6.29: $\{X_n\}$ i.i.d. r.v. and $S_n = \frac{X_1 + \dots + X_n}{\sqrt{n}}$

Prove that $\limsup_{n\to\infty} S_n(\omega) = \infty$ for almost all ω.



Note that $\limsup S_n(\omega) = \infty$
if $\limsup S_n(\omega) > k$ for all k.

$P[\limsup S_n(\omega) > k] = \begin{cases} 0 \\ 1 \end{cases}$ tail event.

Hence we only need to prove that $P[\limsup S_n > k] > 0$.

$P[\limsup_{n\to\infty} S_n > k] = P[\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{m \ge n} [S_m > k]}] = \lim_{n\to\infty} P[\bigcup_{m\ge n} [S_m > k]]$

$\geq \lim_{m\to\infty} P[S_m > k] \xrightarrow{CLT} \int_k^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx > 0.$

By Zero-One Law, we have $P[\limsup S_n > k] = 1$.

Trial Exam 1, Prob 1: X is given by $P[X=0] = \frac{1}{2}$, $P[X=1] = P[X=-1] = \frac{1}{4}$.

a) Find char. fund.

$\varphi_X(t) = E[e^{itX}] = e^{it\cdot 0}\,\frac{1}{2} + e^{it(-1)}\cdot\frac{1}{4} + e^{it\cdot 1}\cdot\frac{1}{4}$

$= \frac{1}{2} + \frac{1}{4}e^{-it} + \frac{1}{4}e^{it} = \frac{1}{2} + \frac{1}{4}(\underbrace{\cos(-t)}_{\cos t} + i\,\cancel{\sin(-t)}_{=-\sin t})$
$+ \frac{1}{4}(\cos t + i\,\cancel{\sin t})$

$= \frac{1}{2} + \frac{1}{2}\cos t = \frac{1}{2}(1 + \cos t)$

b) $\{X_n\}$ independent, with same distribution as X.
$S_n = \frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}}$

Ch.f: $\varphi_{S_n} = E[e^{itS_n}] = E[e^{it\frac{X_1 + \dots + X_n}{\sqrt{n}}}]$

$= E[e^{i\frac{t}{\sqrt{n}}X_1} e^{i\frac{t}{\sqrt{n}}X_2} \dots e^{i\frac{t}{\sqrt{n}}X_n}]$

$\overset{\text{independence}}{=} \underbrace{E[e^{i\frac{t}{\sqrt{n}}X_1}]} E[e^{i\frac{t}{\sqrt{n}}X_2}] \dots E[e^{i\frac{t}{\sqrt{n}}X_n}]$

$= \varphi_X\left(\frac{t}{\sqrt{n}}\right)\cdot \varphi_X\left(\frac{t}{\sqrt{n}}\right) \dots \varphi_X\left(\frac{t}{\sqrt{n}}\right)$

$= \left(\frac{1 + \cos\left(\frac{t}{\sqrt{n}}\right)}{2}\right)^n$

c) Prove that S_n converges in distribution to a normal distribution without using CLT.

Plan: $\varphi_{S_n}(t) \to \varphi(t)$ — ch. f. of a normal distribution $e^{-\frac{\sigma^2 t^2}{2} + i\mu t}$

We get

$\cos x = 1 - \frac{x^2}{2} + o(x^2)$

$\lim_{n\to\infty} \varphi_{S_n}(t) = \lim \left(\frac{1 + \cos\left(\frac{t}{\sqrt{n}}\right)}{2}\right)^n$

$= \lim \left(\frac{1 + \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)}{2}\right)^n$

$= \lim \left(1 \underbrace{- \frac{t^2}{4n} + o\left(\frac{t^2}{n}\right)}_{\frac{z_n}{n}}\right)^n$ $z_n \to -\frac{t^2}{4}$

$= e^{-\frac{t^2}{4}} = e^{-\frac{\frac{1}{2}t^2}{2}}$

$\boxed{\text{Lemma: } z_n \to z, \text{ then } \left(1 + \frac{z_n}{n}\right)^n \to e^z}$

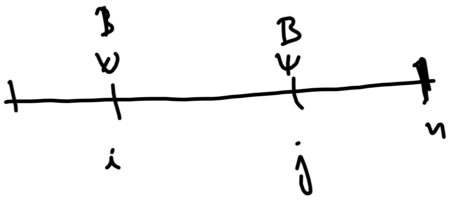
$e^{-\frac{\sigma^2 t}{2} + i\mu t}$ $\mu = 0$, $\sigma^2 = \frac{1}{2}$, $\sigma = \frac{1}{\sqrt{2}}$

Hence by Lévy continuity
$S_n \Rightarrow N(\underset{\mu}{0}, \underset{\sigma^2}{\tfrac{1}{2}})$

Problem 7.16 X_n stochastic process, $B \subseteq \mathbb{R}$.

$T(\omega)$ = the third time $X_n(\omega)$ is in B.

Need to prove that for each n, the set $[T=n] \in \mathcal{F}_n$.



Choose $i < j < n$

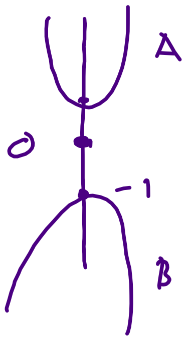
$$A_{i,j} = \{\omega : X_i(\omega) \in B\} \cap \{\omega : X_j(\omega) \in B\} \cap \{\omega : X_n(\omega) \in B\} \cap \{X_1(\omega) \in B^c\} \cap \dots \cap \in \mathcal{F}_n$$

$$[T=n] = \bigcup_{(i,j)} A_{i,j}$$

7.14 T_A, T_B

$$T_{A\cup B} = \min\{T_A, T_B\}$$

Idea $T_{A\cap B} = \cancel{\max\{T_A, T_B\}}$



$T_A = 1$ with prob $1/2$

$T_{A\cap B} = T_\emptyset = \infty$

$T_B = 1$ with prob $1/2$