Characteristic functions

X r.v.

Charact. funct: $\varphi(t) = E[e^{itX}] = \int e^{itx}\, dF(x)$

$$\overset{\text{given density}}{=} \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$$

Th If X has k-th moment, then φ is k-times differentiable and

$$\varphi(t) = \sum_{j=1}^{k} \frac{E[X^j]}{j!} (it)^j + o(t^k)$$

In particular

$$\varphi^{(j)}(0) = i^j E[X^j]$$

Partial converse: If φ is $2k$-times differentiable, then $E[X^{2k}] < \infty$.

Theorem: If X and Y are independent, then

$$\varphi_{X+Y}(t) = \varphi_X(t)\, \varphi_Y(t)$$

independent

Proof: $\varphi_{X+Y}(t) = E[e^{it(X+Y)}] = E[e^{itX} e^{itY}]$

$$= E[e^{itX}]\, E[e^{itY}] = \varphi_X(t)\varphi_Y(t)$$

Joint characteristic function of X and Y:

$$\varphi(s,t) = E[e^{isY + itX}] = E[e^{i(sY + tX)}]$$

Fact: X and Y are independent if and only if:

$$\varphi(s,t) = \varphi_Y(s)\, \varphi_X(t)$$

independent $\Rightarrow$ formula easy

What is the characteristic function of a gaussian random :

If X is $N(0,1)$ - distributed, the density is $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$$\underline{\varphi(t)}=E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx$$

complete square

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+itx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2-2itx}{2}}\,dx$$

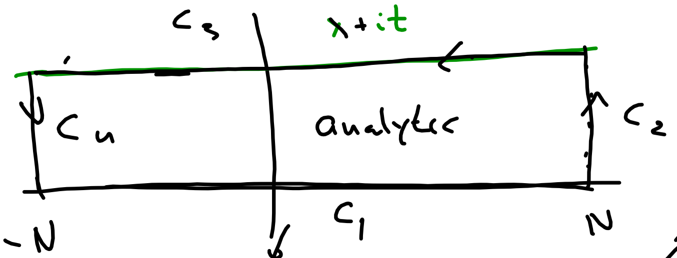
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\overbrace{x^2-2itx+(it)^2}+t^2}{2}}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-it)^2}{2}}\,e^{\frac{t^2}{2}}\,dx$$

$$=\underbrace{\frac{e^{t^2/2}}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-\frac{(x-it)^2}{2}}\,dx}}_{\sqrt{2\pi}}$$

$e^{-\frac{z^2}{2}}$ for $z=x-it$

Would have liked to put $u=x-it$



$$\int_{-\infty}^{\infty} e^{-\frac{(x-it)^2}{2}}\,dx = \lim_{N\to\infty}\int_{-N}^{N} e^{-\frac{(x-it)^2}{2}}\,dx$$

$$0=\int_C e^{-\frac{z^2}{2}}\,dz = \int_{C_1} + \underbrace{\int_{C_2}}_{\to 0} + \int_{C_3} + \underbrace{\int_{C_4}}_{\to 0}$$

Conclusion: $\int_{-\infty}^{\infty} e^{-\frac{(x-it)^2}{2}}\,dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$

Hence $\varphi(t) = \underline{e^{t^2/2}}$.

What if $Y = N(\mu, \sigma^2)$?

$Y = \sigma X + \mu$ is $N(\mu, \sigma^2)$

$N(0,1)$

$$\varphi_Y(t) = E\left[e^{itY}\right] = E\left[e^{it(\sigma X + \mu)}\right]$$

$$= E\left[e^{it\mu} \cdot e^{it\sigma X}\right] = e^{it\mu} E\left[e^{i(t\sigma)X}\right]$$

$$= e^{it\mu} \underbrace{\varphi_X(t\sigma)}_{e^{\frac{(t\sigma)^2}{2}}} = e^{it\mu} e^{\frac{(\sigma t)^2}{2}}$$

$$= e^{i\mu t + \frac{\sigma^2 t^2}{2}}$$

<u>Next time:</u> The inversion problem

<u>What we know:</u> Given the distribution F, we can compute the characteristic function φ, $\varphi(t) = \int e^{itx}\, dF(x)$.

<u>Inverse problem:</u> If we know φ, can we find F?

Problems:

pp 126–128: 4.11, 4.13, 4.23, 4.28

Assign. 2019, problem 1

4.11: $X_n \to \infty$ a.s. $\iff P[X_n < M \text{ i.o}] = 0$

Reminder: $X_n(\omega) \to \infty$ if for all M there is a N such that $X_n(\omega) \geq M$ for all $n \geq N$.

Assume $X_n \to \infty$ a.s. This means that there is only a set Ω_0 of measure 0 such that $X_n \not\to \infty$.

But
$$\underbrace{[X_n < M \text{ i.o}]}_{X_n \not\to \infty} \subseteq \Omega_0$$
and since $P(\Omega_0) = 0$, $P[X_n < M \text{ i.o}] = 0$.

Assume that X_n does not go to ∞ a.s. Then there must be a set Ω_1 of prob $P(\Omega_1) = a > 0$, such that $X_n(\omega) \not\to \infty$ when $\omega \in \Omega_1$.

But $\Omega_1 \subset \bigcup_{M \in \mathbb{N}} [X_n < M \text{ i.o}]$

$$0 < a = P(\Omega_1) \leq P\Big(\underbrace{\bigcup_{M \in \mathbb{N}}}_{\text{increasing union}} [X_n < M \text{ i.o}]\Big)$$

$$\overset{\text{cont. meas}}{=} \lim_{M \to \infty} P[X_n < M \text{ i.o}]$$

Hence $P[X_n < M \text{ i.o}]$ has to be positive for sufficiently large M.

4.23: Assume that $\{X_n\}$ is a sequence of <u>independent</u> random variables converging pointwise to X. Show that X is a.s. constant.

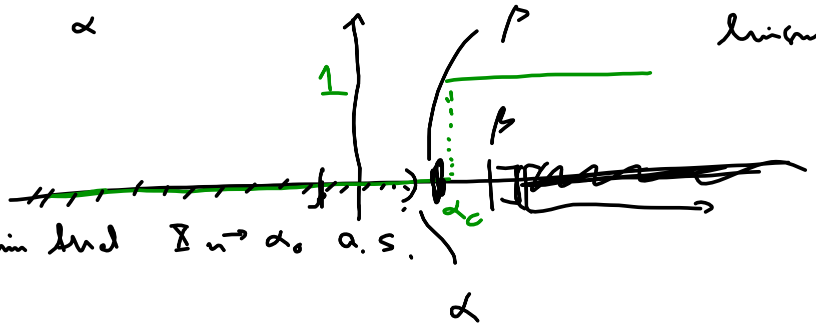
Fix $\alpha \in \mathbb{R}$ and look at the sets

$$A_n^\alpha = \{\omega : X_n(\omega) \le \alpha\}$$

According to Borel Cantelli (II), we know that

$$\underbrace{P[\limsup A_n^\alpha]}_{\alpha} \text{ is either } 0 \text{ or } 1.$$

$\limsup A_n^\beta = \emptyset$



Claim that $X_n \to \alpha_0$ a.s.

Assignment problem: Assume that $X_1, X_2, \ldots, X_n$ are independent random variables with 4th moments and mean 0.

$$E\Big[\Big(\underline{\sum_{i=1}^{n} X_i}\Big)^4\Big] = \sum_{r=1}^{n} E[X_r^4] + 6\sum_{p<q} E[X_p^2 X_q^2]$$

$$\because\ E\Big[\Big(\sum_{i=1}^{n} X_i\Big)^4\Big] = \sum_{i_1, i_2, i_3, i_4} E[\underline{X_{i_1}}\ \underline{X_{i_2}}\ \underline{X_{i_3}}\ \underline{X_{i_4}}]$$

$$= \sum_{r=1}^{n} E[X_r^4] + \sum_{p<q} 6 E[\underbrace{X_p^2 X_q^2}_{p \neq q}] \qquad \begin{matrix}(p, q) \\ p < q\end{matrix}$$

$$= \sum_{r=1}^{n} E[X_r^4] + 6\sum_{p<q} E[X_p^2] E[X_q^2]$$

4.13 Show that $\{X_n\}$ converges to X in probability iff every subsequence $\{X_{n_k}\}$ has a further subsequence which converges to X a.s.

Assume first that $X_n \to X$ in prob. Then any subsequence $\{X_{n_k}\}$ converges to X in prob. and hence has a further subsequence converging to X a.s.

Assume that X_n does not converge to X in prob., then has to be an $\varepsilon > 0$ s.t

$$P[\,|X - X_n| \geq \varepsilon\,] \not\to 0.$$

This means that there is a $\eta > 0$ such that

$$P[\,|X - X_n| \geq \varepsilon\,] \geq \eta \text{ for infinitely many } n.$$

Take these n's and turn them into a subsequence $\{X_{n_k}\}$. For all k

$$P[\,|X - X_{n_k}| \geq \varepsilon\,] \geq \eta$$

So for any subsequence $\{X_{n_{k_\ell}}\}$ of $\{X_{n_k}\}$,

$$P[\,|X - X_{n_{k_\ell}}| \geq \varepsilon\,] \geq \eta.$$

Hence $X_{n_{k_\ell}}$ does not go to X in prob. and hence $X_{n_{k_\ell}}$ does not converge to X a.s. QED