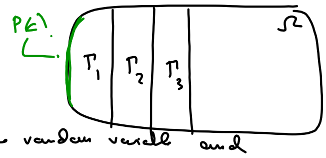
Expectations of discrete r.v.

Recall: $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a quasi-partition of Ω if all Γ_n's are events, $\Gamma_i\cap\Gamma_j=\emptyset$ for $i\neq j$, and $P(\Omega\setminus\bigcup_{n\in\mathbb{N}}\Gamma_n)=0$



Recall: If X is a discrete random variable and $\{\Gamma_n\}$ is a quasi-partition such that X is constant on each Γ_n with $X=x_n$. Then X is integrable iff $\sum_{n=1}^{\infty}|x_n|P(\Gamma_n)<\infty$ and then $E[X]=\sum_{n=1}^{\infty}x_nP(\Gamma_n)$

Theorem: Let $X,Y:\Omega\to\mathbb{R}$ be two discrete random variables. Then

(i) X is integrable iff $|X|$ is integrable.

(ii) If X,Y are integrable and $a,b\in\mathbb{R}$, then $aX+bY$ is integrable and

$$E[aX+bY]=aE[X]+bE[Y]$$

(iii) If $|X|\le|Y|$ and Y is integrable, then X is integrable.

(iv) If X,Y are integrable and $X\le Y$, then

$$E[X]\le E[Y]$$

(v) If X is integrable, then $E[X]\le E[|X|]$

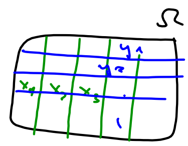
Proof (i) We have
- X integrable $\Leftrightarrow \sum_{i=1}^{\infty}|x_i|P(\Gamma_i)<\infty$
- $|X|$ integrable $\Leftrightarrow \sum_{i=1}^{\infty}|x_i|P(\Gamma_i)<\infty$

(ii) Let $\{x_i\}$ be the points that X takes with prob >0

— " — $\{y_j\}$ — " — Y — " —

Define a quasi-partition Γ_{ij}:

$$\Gamma_{ij}=\{\omega: X(\omega)=x_i \text{ and } Y(\omega)=y_j\}$$



Assume X and Y are integrable. Then $aX+bY$ is integrable because:

$$\sum_{i,j}|ax_i+by_j|P(\Gamma_{ij})\le\sum_{i,j}(|a||x_i|+|b||y_j|)P(\Gamma_{ij})$$

$$=|a|\overbrace{\sum_{i,j}|x_i|P(\Gamma_{ij})}^{<\infty}+|b|\overbrace{\sum_{i,j}|y_j|P(\Gamma_{ij})}^{<\infty}<\infty.$$

Also

$$E[aX+bY]=\sum_{i,j}(ax_i+by_j)P(\Gamma_{ij})$$

$$=a\overbrace{\sum_{i,j}x_iP(\Gamma_{ij})}+b\overbrace{\sum y_jP(\Gamma_{ij})}=aE[X]+bE[Y].$$

(iii) Assume $|X|\le|Y|$ and Y is integrable:

$$\sum_{i,j}|x_i|P(\Gamma_{ij})\le\sum_{i,j}|y_j|P(\Gamma_{ij})<\infty$$

Hence X is integrable.

(iv) Assume X,Y integrable and $X\le Y$. Must show $E[X]\le E[Y]$

$$E[X]=\sum_{i,j}x_iP(\Gamma_{ij})\le\sum_{i,j}y_jP(\Gamma_{ij})=E[Y].$$

(v) Assume X integrable. Since $X\le|X|$, (iv) gives us

$$E[X]\le E[|X|].$$

Expectation for general random variables

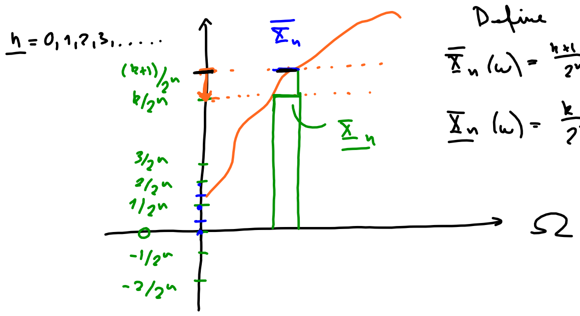
X r.v. **Problem:** How to define $E[X]$?

Idea: Find sequences $\{\overline{X}_n\}$ and $\{\underline{X}_n\}$ of discrete random variables such

$$\underline{X}_n \le X \le \overline{X}_n$$

$$\underline{X}_n \nearrow X, \quad \overline{X}_n \searrow X \quad \text{and} \quad \lim_{n\to\infty} E[\overline{X}_n] = \lim_{n\to\infty} E[\underline{X}_n]$$

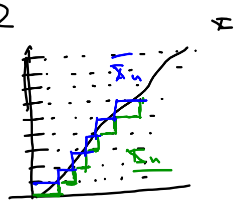
Define $E[X] = \lim E[\overline{X}_n] = \lim E[\underline{X}_n]$



Define

$$\overline{X}_n(\omega) = \frac{k+1}{2^n} \text{ if } \frac{k}{2^n} < X(\omega) \le \frac{k+1}{2^n}$$

$$\underline{X}_n(\omega) = \frac{k}{2^n} \text{ if } \frac{k}{2^n} < X(\omega) \le \frac{k+1}{2^n}$$



Observations:

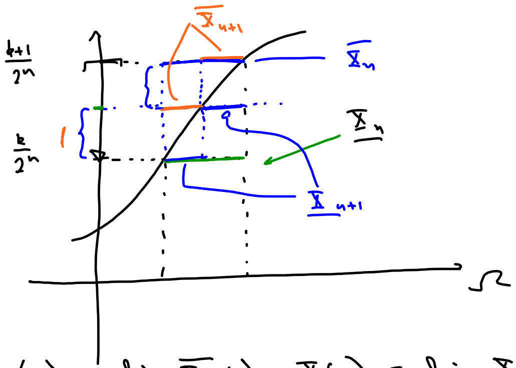
(i) $\overline{X}_n$ and $\underline{X}_n$ are random variable and

$$\underline{X}_n(\omega) < X(\omega) \le \overline{X}_n(\omega)$$

and $\overline{X}_n(\omega) - \underline{X}_n(\omega) = \frac{1}{2^n}$.

(ii) The sequence $\{\overline{X}_n\}$ is decreasing and $\{\underline{X}_n\}$ is increasing

(i.e. $\overline{X}_{n+1}(\omega) \le \overline{X}_n(\omega)$ and $\underline{X}_{n+1}(\omega) \ge \underline{X}_n(\omega)$)

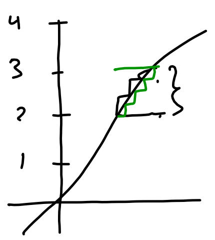


(iii) $\lim_{n\to\infty} \overline{X}_n(\omega) \searrow = X(\omega) = \lim_{n\to\infty} \underline{X}_n(\omega) \nearrow$

(iv) The approximations $\overline{X}_n$ and $\underline{X}_n$ are either all integrable or all nonintegrable.

Proof: It suffices to prove that if Y and Z are two approximations and Y is integrable, then Z is also integrable.

Since Y and Z are approximations, $|Y - Z| \le 1$.

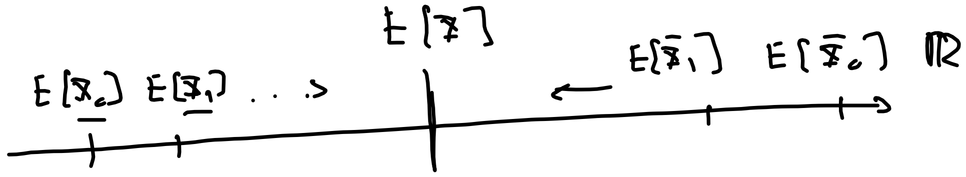


But then $\underset{\text{integrable.}}{|Z|} \le \underbrace{|Y| + 1}_{\text{integrable}}$

We have $E[\overline{X}_n]$ is decreasing

— " — $E[\underline{X}_n]$ is increasing

$$E[\overline{X}_n]-E[\underline{X}_n] = E[\overline{X}_n-\underline{X}_n] \le E[\overbrace{|\overline{X}_n-\underline{X}_n|}^{\frac{1}{2^n}}] = \frac{1}{2^n} \to 0$$



We have shown that $\lim_{n\to\infty} E[\underline{X}_n] = \lim E[\overline{X}_n]$

<u>Definition</u>: Let X be a random variable. We say that X is <u>integrable</u> if $\overline{X}_0$ is integrable (which is the same as saying $\overline{X}_n, \underline{X}_n$ are integrable for all n).
In this case, we define the expectation as

$$E[X] = \lim_{n\to\infty} E[\overline{X}_n] = \lim_{n\to\infty} E[\underline{X}_n].$$

<u>Theorem</u>: Let $X, Y: \Omega \to \mathbb{R}$ be random variables. Then

(i) If $X = Y$ a.e. (i.e. $P\{\omega : X(\omega) \ne Y(\omega)\} = 0$) and Y is integrable, then X is integrable and $E[X] = E[Y]$.

(ii) If $|X| \le |Y|$ and Y is integrable, then X is integrable.

(iii) If X is integrable and $a \in \mathbb{R}$, then aX is integrable and $E[aX] = aE[X]$.

(iv) If X, Y are integrable, then $X+Y$ is integrable and
$E[X+Y] = E[X] + E[Y]$

(v) If $X \ge 0$ a.s., then $E[X] \ge 0$

(vi) If X, Y are integrable and $X \le Y$, then $E[X] \le E[Y]$.

Proof: (i) $X = Y$ a.e., Y integrable.

X is integrable $\Leftrightarrow$ $\overline{X}_n$ is integrable $\Leftrightarrow$ $\sum \left|\frac{k+1}{2^n}\right| P\left[\underbrace{\frac{k}{2^n} < X(\omega) \le \frac{k+1}{2^n}}_{Y}\right] < \infty$

$\Leftrightarrow \sum \left|\frac{k+1}{2^n}\right| P\left[\frac{k}{2^n} < Y(\omega) \le \frac{k+1}{2^n}\right] < \infty \Leftrightarrow Y$ is integrable.

$$E[\overline{X}_n] = \sum \frac{k+1}{2^n} P\left[\frac{k}{2^n} < X(\omega) \le \frac{k+1}{2^n}\right]$$

$$= \sum \frac{k+1}{2^n} P\left[\frac{k}{2^n} < Y(\omega) \le \frac{k+1}{2^n}\right] = E[\overline{Y}_n]$$

Hence $E[X] = \lim_{n\to\infty} E[\overline{X}_n] = \lim_{n\to\infty} E[\overline{Y}_n] = E[Y]$

(ii) Assume $|X| \le |Y|$ and Y is integrable. Show that X is integrable. We have

$$|\overline{X}_0| - 1 \le |X| \le |Y| \le |\overline{Y}_0| + 1$$

Hence $|\overline{X}_0| \le \underbrace{|\overline{Y}_0| + 2}_{\text{integrable}}$, and hence $\overline{X}_0$ is integrable By definition X is integrable.

(iii) X is integrable. Want to show that aX is integrable and that $E[aX] = aE[X]$.

Note that: $|aX| = |a||X| \le \underbrace{|a|(|\overline{X}_0| + 1)}_{\text{integrable}}$

Hence aX is integrable by (ii).

Let $Z = aX$. Then

$$|\overline{Z}_n - a\overline{X}_n| = |(\overline{Z}_n - Z) + (aX - a\overline{X}_n)|$$

$$\le |\overline{Z}_n - Z| + |a||X - \overline{X}_n| \le \frac{1}{2^n} + |a|\frac{1}{2^n} = \frac{|a|+1}{2^n}.$$

Hence

$$|\underbrace{E[\overline{Z}_n]}_{\to E[Z]} - a\underbrace{E[\overline{X}_n]}_{\to E[X]}| = |E[\overline{Z}_n - a\overline{X}_n]|$$

$$\le E[|\overline{Z}_n - a\overline{X}_n|] \le \frac{|a|+1}{2^n} \to 0$$

The two limits have to be the same, i.e.

$$E[Z] = aE[X].$$