Expectations

$E[X] = \lim_{n\to\infty} E[\overline{X}_n]$, X is integrable iff $\overline{X}_0$ is integrable.

Thm: Assume $X, Y : \Omega \to \mathbb{R}$ are random variables.

(i) If X, Y are integrable, then $X+Y$ is integrable

$$E[X+Y] = E[X] + E[Y]$$

(ii) If $X \geq c$ a.s, then $E[X] \geq c$

(iii) If X, Y are integrable and $X \leq Y$, then $E[X] \leq E[Y]$.

Proof (i) $|X+Y| \leq |X| + |Y| \leq |\overline{X}_0| + 1 + |\overline{Y}_0| + 1$

$$= \underbrace{|\overline{X}_0| + |\overline{Y}_0| + 2}_{\text{integrable as a sum of discrete, integrable r.v.}}$$

Hence $X+Y$ is integrable.

Let $Z = X+Y$: Then

$$|\overline{Z}_n - (\overline{X}_n + \overline{Y}_n)| \leq |\overline{Z}_n - Z + (X+Y) - (\overline{X}_n + \overline{Y}_n)|$$

$$\leq |\overline{Z}_n - Z| + |X - \overline{X}_n| + |Y - \overline{Y}_n| \leq \frac{3}{2^n}.$$

$\quad\leq \frac{1}{2^n} \qquad \leq 1/2^n \qquad \leq 1/2^n$

$E[Z] \qquad (E[X] + E[Y])$

$$|E[\overline{Z}_n] - (E[\overline{X}_n] + E[\overline{Y}_n])| = |E[\overline{Z}_n - (\overline{X}_n + \overline{Y}_n)]|$$

$$\leq E[|\overline{Z}_n - (\overline{X}_n + \overline{Y}_n)|] \leq \frac{3}{2^n} \to 0$$

hence $E[Z] = E(X) + E(Y)$.

(ii) If $X \geq 0$ a.s, then $E[X] \geq 0$.

$$E[\overline{X}_n] = \sum_{k\in\mathbb{Z}} \frac{k+1}{2^n} P\left[\frac{k}{2^n} < X(\omega) \leq \frac{k+1}{2^n}\right] \geq 0$$

But

$$E[X] = \lim_{n\to\infty} E[\overline{X}_n] \geq 0.$$

(iii) If X, Y integrable and $X \leq Y$, then $E[X] \leq E[Y]$.

$$E[Y] = E[X + (Y-X)] = E[X] + \underset{\substack{\vee\!\vert \\ 0}}{E[Y-X]} \geq E[X].$$

Theorem: Assume that $X, Y : \Omega \to \mathbb{R}$ are integrable and independent random variables. Then XY is integrable and

$$E[XY] = E[X]\,E[Y].$$

Proof: Assume first that X and Y are discrete taking values x_i and y_j.

First check integrability:

$$\sum_{i,j} |x_i y_j| \, \underbrace{P[X = x_i \text{ and } Y = y_j]}_{\text{independence}}$$

$$= \sum_{i,j} |x_i| |y_j| \, P[X = x_i] P[Y = y_j]$$

$$= \Big(\sum_i |x_i| P[X = x_i] \Big) \Big(\sum |y_j| P[Y = y_j] \Big) < \infty.$$

Now: $E[XY] = \sum x_i y_j P[X = x_i \text{ and } Y = y_j]$

$$= \sum x_i y_j P[X = x_i] P[Y = y_j]$$

$$= \Big(\sum x_i P[X = x_i] \Big) \Big(\sum y_j P[Y = y_j] \Big) = E[X]\,E[Y]$$

Key estimate:

$$E[|XY - \bar{X}_N \bar{Y}_N|] = E[|(XY - X\bar{Y}_N) + (X\bar{Y}_N - \bar{X}_N \bar{Y}_N)|]$$

$$\le E[|X| \underbrace{|Y - \bar{Y}_N|}_{\le \frac{1}{2^N}}] + E[|\bar{Y}_N| \underbrace{|X - \bar{X}_N|}_{\frac{1}{2^N}}]$$

$$\le \frac{1}{2^N} E[|X|] + \frac{1}{2^N} E[|\bar{Y}_N|]$$

$$\le \frac{1}{2^N} \big[E[|\bar{X}_0|] + 1 + E[|\bar{Y}_0|] + 1 \big] \to 0.$$

Check that XY is integrable

$$XY = \underbrace{\bar{X}_N \bar{Y}_N}_{\text{integrable}} + \underbrace{(XY - \bar{X}_N \bar{Y}_N)}_{\text{integrable by estimate.}} \Rightarrow XY \text{ is integrable.}$$

$$\Big| E[XY] - \underbrace{E[\overset{E[X]}{\bar{X}_N}] E[\overset{E[Y]}{\bar{Y}_N}]}_{(E[\bar{X}_N \bar{Y}_N])} \Big| \le E[|XY - \bar{X}_N \bar{Y}_N|] \to 0$$

by the estimate

Hence $E[XY] = E[X]\,E[Y]$.

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p 46-48: 2.4, 2.7, 2.10, <u>2.16</u>, <u>2.17</u>

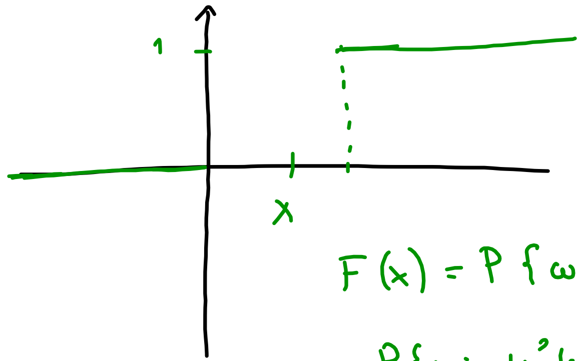
49: <u>2.20</u>

53: 2.22, 2.30

<u>Extra</u>: Jensens Hölder.

2.4 U uniform distribution on $[0,1]$

$X = U^2$ (taking values between 0 and 1)

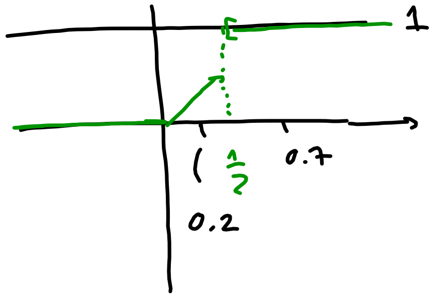


$$F(x) = P\{\omega : X(\omega) \le x\}$$

$$= P\{\omega : U^2(\omega) \le x\} = P\{\omega : U(\omega) \le \sqrt{x}\} = \underline{\underline{\sqrt{x}}}$$

2.7 Distribution function F of X:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < \frac{1}{2} \\ 1 & \text{if } x \ge 1/2 \end{cases}$$



$$P\{X < \tfrac{1}{2}\} = F(\tfrac{1}{2}-) = \tfrac{1}{2}.$$

$$P\{0.2 \le X \le 0.7\} = F(0.7) - F(0.2-) = 1 - 0.2 = \underline{0.8}$$

$$P\{X = \tfrac{1}{2}\} = F(\tfrac{1}{2}) - F(\tfrac{1}{2}-) = 1 - \tfrac{1}{2} = \underline{\underline{\tfrac{1}{2}}}$$

2.10 5 boys 5 girls in a queue.

X = the position of the first girl.

$$P[X=1] = \frac{1}{2}.$$

$$P[X=2] = \frac{1}{2} \cdot \frac{5}{9}$$

$$P[X=3] = \frac{1}{2} \cdot \frac{4}{9} \cdot \frac{5}{8}$$

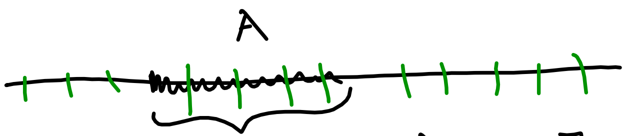
$$P[X=4] = \frac{1}{2} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{5}{7}$$

$$P[X=5] = \frac{1}{2} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} \cdot \frac{5}{6}$$

$$P[X=6] = \frac{1}{2} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} \cdot \frac{1}{6} \cdot 1$$

How to get a measure:

$\{a_k\}_{k=1}^{n}$ with prob $\{p_k\}_{k=1}^{n}$

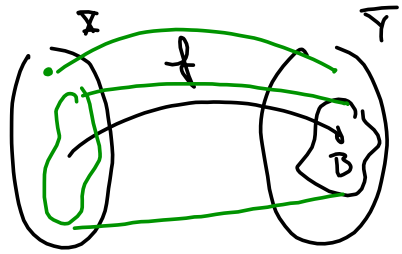


$$\mu(A) = P[X \in A]$$

$$= \sum_{a_k \in A} p_k$$

Extra problem: $f: X \to Y$

Inverse image: $B \subseteq Y$, then $f^{-1}(B) = \{x \in X : f(x) \in B\}$



(i) $f^{-1}(\phi) = \phi$.

Proof: $f^{-1}(\phi) = \{x \in X : f(x) \in \phi\} = \phi$

(ii) $f^{-1}(\underbrace{B^c}_{B^c = Y - B}) = \underbrace{(f^{-1}(B))^c}_{X \setminus f^{-1}(B)}$

Proof: $x \in f^{-1}(B^c) \Leftrightarrow f(x) \in B^c \Leftrightarrow f(x) \notin B$

$\Leftrightarrow x \notin f^{-1}(B) \Leftrightarrow x \in (f^{-1}(B))^c$

(iii) $f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$

Proof: $x \in f^{-1}\left(\bigcup_{i \in I} B_i\right) \Leftrightarrow f(x) \in \bigcup_{i \in I} B_i$

$\Leftrightarrow f(x) \in B_i$ for at least one i

$\Leftrightarrow x \in f^{-1}(B_i)$ — " —

$\Leftrightarrow x \in \bigcup_{i \in I} f^{-1}(B_i)$

(iv) $f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$

$x \in f^{-1}\left(\bigcap_{i \in I} B_i\right) \Leftrightarrow f(x) \in \bigcap_{i \in I} B_i \Leftrightarrow f(x) \in B_i$ for all $i \in I$

$\Leftrightarrow x \in f^{-1}(B_i)$ for all $i \in I \Leftrightarrow x \in \bigcap_{i \in I} f^{-1}(B_i)$

2.30 Toss a fair coin four times.

X = number of heads - number of tails.

H	T	X
4	0	4
3	1	2
2	2	0
1	3	-2
0	4	-4

$P[X=4] = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{16}$

$P[X=2] = 4\cdot\frac{1}{16} = \frac{1}{4}$

$P[X=0] = 6\cdot\frac{1}{16} = \frac{3}{8}$

$P[X=-2] = \frac{1}{4}$

$P[X=-4] = \frac{1}{16}$

HHTT

HTHT

$\binom{4}{2} = \frac{4\cdot 3}{1\cdot 2} = 6$

2.21 X, Y are independent, $\frac{1}{2}$ Bernoulli

X	Y	Prob	$Z=(X-Y)^2$
1	1	1/4	0
1	0	1/4	1
0	1	1/4	1
0	0	1/4	0

(X,Z) independent

(Y,Z) — " —

(X,Y) — " —

X, Y, Z are not independent

$P[X=1, Y=1, Z=1] = 0$ $\neq$ not independent

$P[X=1]P[Y=1]P[Z=1] = \frac{1}{8}$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$