Martingales

Observation: X, Y are integrable random variables, Y is $\mathcal{G}$-measurable where $\mathcal{G} \subseteq \mathcal{F}$. If we want to prove that

(*) $\quad E[X|\mathcal{G}] \geq Y$ a.s.

then it suffices to prove that

(**) $\quad \int_\Lambda X \, dP \geq \int_\Lambda Y \, dP$ for all $\Lambda \in \mathcal{G}$.

(and vice versa).

Proof: Assume (**) holds. Then

$$\int_\Lambda E[X|\mathcal{G}] \, dP = \cancel{\int_\Lambda X \, dP} \geq \int_\Lambda Y \, dP$$

and hence

$$\int_\Lambda (E[X|\mathcal{G}] - Y) \, dP \geq 0 \text{ for all } \Lambda \in \mathcal{G}$$

Choose $\Lambda = \{\omega : Y > E[X|\mathcal{G}]\}$. Then

$$\int_{[Y > E[X|\mathcal{G}]]} \overbrace{(E[X|\mathcal{G}] - Y)}^{<0} \, dP \geq 0, \text{ hence } P[Y > E[X|\mathcal{G}]] = 0$$

Definition: Assume that $\{\mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a filtration and that $X = \{X_n\}_{n \in \mathbb{N}_0}$ is a process. If the following conditions are satisfied

(i) Each X_n is integrable

(ii) X is adapted to $\{\mathcal{F}_n\}$ (i.e. X_n is $\mathcal{F}_n$-measurable for each n)

(iii) Whenever $t \geq s$, then $E[X_t | \mathcal{F}_s] = X_s$ a.s.

then X is said to be a <u>martingale</u> w.r.t. $\{\mathcal{F}_n\}$.

If we replace (iii) by

(iv) $E[X_t | \mathcal{F}_s] \geq X_s$ a.s

then X is a <u>submartingale</u>

and if we replace (iv) by

(v) $E[X_t | \mathcal{F}_s] \leq X_s$ a.s

then X is a <u>supermartingale</u>

Observations: (i) If X is a submartingale, then $-X$ is a supermartingale and vice versa.

(ii) If X is both a sub- and supermartingale, then X is a martingale.

Prop: Assume that X is a sub- / super- / ordinary martingale w.r.t a filtration $\{\mathcal{F}_t\}$. Assume that $\{\mathcal{G}_t\}$ is a coarser filtration (i.e. $\mathcal{G}_t \subseteq \mathcal{F}_t$) such that X is $\mathcal{G}_t$-adapted. Then X is also a $\{\mathcal{G}_t\}$- sub / super / ordinary martingale.

Proof (for submartingales) It suffices to prove that $E[X_t | \mathcal{G}_s] \geq X_s$, which means that we need to prove that

$$\int_\Lambda X_t \, dP \geq \int_\Lambda X_s \, dP \text{ for all } \Lambda \in \mathcal{G}_s.$$

Since $\{X_t\}$ is an $\{\mathcal{F}_t\}$-martingale, this holds for all $\Lambda \in \mathcal{F}_s$, and since $\mathcal{G}_s \subseteq \mathcal{F}_s$, it holds for all $\Lambda \in \mathcal{G}_s$

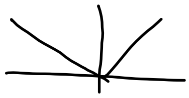
Theorem (i) Assume that X is a martingale and φ is a convex function. If $\varphi(X_t)$ is integrable for each t, then $\{\varphi(X_t)\}_{t\in\mathbb{N}_0}$ is a submartingale.

(ii) Assume that X is a submartingale and that φ is an increasing, convex function. If $\varphi(X_t)$ is integrable for each t, then $\{\varphi(X_t)\}_{t\in\mathbb{N}_0}$ is a submartingale.

Proof (ii)

$$\underbrace{\varphi(X_s) \overset{\text{increasing}}{\leq} \varphi(E[X_t|\mathcal{F}_s]) \overset{\text{J.in.}}{\leq} E[\varphi(X_t)|\mathcal{F}_s]}$$

hence $\varphi(X_t)$ is a submartingale.



Example: If X is a martingale, then $|X|$ is a submartingale. If $p\geq 1$ and $|X_t|^p$ is integrable, then $\{|X|_t^p\}$ is a submartingale.

Def: If $X=\{X_n\}_{n\in\mathbb{N}_0}$ is a stochastic process, then define

$$\Delta X_n = X_{n+1}-X_n.$$

Note that if X_n is $\{\mathcal{F}_n\}$ adapted, then ΔX_n is $\mathcal{F}_{n+1}$ measurable, and that

$$X_n = X_0 + \sum_{j=0}^{n-1}\Delta X_j$$

Prop: Assume that $\{X_n\}$ is integrable and $\{\mathcal{F}_n\}$-adapted. Then X_n is a submartingale if and only if

$$(*) \qquad E[\Delta X_n|\mathcal{F}_n]\geq 0 \text{ for all } n.$$

(Similarly, $E[\Delta X_n|\mathcal{F}_n]\leq 0$ for supermartingales and $E[\Delta X_n|\mathcal{F}_n]=0$ for martingales)

Proof (for submartingales): Assume that X is a submartingale. Then

$$E[\Delta X_n|\mathcal{F}_n] = E[X_{n+1}-X_n|\mathcal{F}_n] = E[X_{n+1}|\mathcal{F}_n]-E[X_n|\mathcal{F}_n]$$

$$= \underbrace{E[X_{n+1}|\mathcal{F}_n]}-\underbrace{X_n} \geq 0 \text{ by the submartingale property.}$$

Assume that X satisfies $(*)$. Then if $m>n$

$$E[X_m|\mathcal{F}_n] - \underbrace{X_n}_{E[X_n|\mathcal{F}_n]} = E[X_m-X_n|\mathcal{F}_n]$$

$$= E\left[\sum_{j=n}^{m-1}\Delta X_j\Big|\mathcal{F}_n\right] = \sum_{j=n}^{m-1}E[\Delta X_j|\mathcal{F}_n]$$

$$= \sum_{j=n}^{m-1}\underbrace{E[\underbrace{E[\Delta X_j|\mathcal{F}_j]}_{\geq 0}|\mathcal{F}_n]}_{\geq 0} \geq 0$$

hence $E[X_m|\mathcal{F}_n]\geq X_n$ a.s. and hence X is a submartingale.

<u>Doob's Decomposition Theorem</u>: Assume that $X = \{X_n\}_{n\in\mathbb{N}_0}$ is a submartingale. Then

$$\underline{X_n = M_n + A_n}$$

where

(i) M_n is a martingale

(ii) A_n is $\mathcal{F}_{n-1}$-measurable

(iii) $A_0 = 0$ and $A_{n+1} \geq A_n$ a.e.

<u>Proof</u>: Define $A_0 = 0$ and $\Delta A_n = E[\Delta X_n \mid \mathcal{F}_n]$. Then

$$A_n = \sum_{j=0}^{n-1} \Delta A_j = \sum_{j=1}^{n-1} \underbrace{E[\Delta X_j \mid \mathcal{F}_j]}_{\mathcal{F}_j\text{-measurable}}$$ is $\mathcal{F}_{n-1}$-measurable and <u>integrable</u>.

A_n is increasing since

$$A_{n+1} - A_n = \Delta A_n = \underline{E[\Delta X_n \mid \mathcal{F}_n]} \geq 0.$$

Now define $M_n = X_n - A_n$ (hence $X_n = M_n + A_n$). It remains to prove that M_n is a martingale. Note M_n is integrable since X_n and A_n are, and M_n is $\mathcal{F}_n$-measurable since X_n and A_n are. Need to check

$$E[\Delta M_n \mid \mathcal{F}_n] = E[\Delta X_n - \underline{\Delta A_n} \mid \mathcal{F}_n]$$

$$= E[\Delta X_n \mid \mathcal{F}_n] - E[E[\Delta X_n \mid \mathcal{F}_n] \mid \mathcal{F}_n]$$

$$= E[\Delta X_n \mid \mathcal{F}_n] - E[\Delta X_n \mid \mathcal{F}_n] = 0$$

and hence M is a martingale.

Martingales and stopping times

Reminder: T stopping time $\left.\begin{array}{l}[T \leq n] \in \mathcal{F}_n \\ [T = n] \in \mathcal{F}_n\end{array}\right\}$ for all n.

$\Lambda \in \mathcal{F}_T \Leftrightarrow \begin{array}{l}\Lambda \cap [T \leq n] \in \mathcal{F}_n \\ \Lambda \cap [T = n] \in \mathcal{F}_n\end{array}$ for all n.

Proposition: Assume that $X = \{X_n\}_{n \in \mathbb{N}_0}$ is an $\{\mathcal{F}_n\}$-$\overset{\text{sub}}{\vee}$martingale, and let T be a stopping time. Then the stopped process

$\{X_{T \wedge n}\}_{n \in \mathbb{N}}$ is a submartingale wrt to $\{\mathcal{F}_n\}$ and also wrt to $\{\mathcal{F}_{T \wedge n}\}_{n \in \mathbb{N}}$.

Proof: We need to prove that $X_{T \wedge n}$ is integrable and that

(*) $\quad E[\Delta X_{T \wedge n} \mid \mathcal{F}_n] \geq 0$.

For integrability, note that

$$|X_{T \wedge n}| \leq \underbrace{|X_1| + |X_2| + \dots + |X_n|}_{\text{is integrable}}$$

and $|X_{T \wedge n}|$ is integrable. Need to show

$$\int_\Lambda \Delta X_{T \wedge n} \, dP \geq \int_\Lambda 0 \, dP = 0 \text{ for all } \Lambda \in \mathcal{F}_n.$$

But

$$\int_\Lambda \Delta X_{T \wedge n} \, dP = \int_{\underbrace{[n < T]}_{\substack{[T \leq n]^c \\ \wedge \mathcal{F}_n}}} \Delta X_n \, dP + \int_{[n \geq T]} 0 \, dP$$

$$= \int_{[n < T]} \underbrace{E[\Delta X_n \mid \mathcal{F}_n]}_{\substack{\vee\!| \\ 0}} \, dP + 0 \geq 0.$$

<u>Theorem</u>: Assume that $X = \{X_n\}_{n \in \mathbb{N}_0}$ is a submartingale and let S, T be two bounded stopping times $S \leq T$. Then

$$E[X_T \mid \mathcal{F}_S] \geq X_S \quad \text{a.e.}$$

(and similarly for supermartingales and martingales).

<u>Proof</u>: $X_T = X_S + \underline{\sum_{j=S}^{T-1} \Delta X_j} = X_S + \sum_{j=0}^{N} 1\!\!1_{\{S \leq j < T\}} \Delta X_j$

where $N \geq T, S$. Then

$$E[X_T \mid \mathcal{F}_S] = E\Big[X_S + \sum_{j=0}^{N} 1\!\!1_{\{S \leq j < T\}} \Delta X_j \mid \mathcal{F}_S\Big]$$

$$= \underbrace{E[X_T \mid \mathcal{F}_S]}_{X_S} + \sum_{j=0}^{N} E\Big[\overbrace{1\!\!1_{\underset{\mathcal{F}_j}{\{S \leq j\}} \cap \underbrace{\{j < T\}}_{\{T \leq j\}^c}} \Delta X_j}\mid \mathcal{F}_S\Big]$$

$$= X_S + \sum_{j=0}^{N} E\Big[\underbrace{E[1\!\!1_{\{S \leq j\} \cap \{j < T\}} \Delta X_j \mid \mathcal{F}_j]}\mid \mathcal{F}_S\Big]$$

$$\underbrace{E\Big[1\!\!1_{\{S \leq j\} \cap \{j < T\}} E[\Delta X_j \mid \mathcal{F}_j] \mid \mathcal{F}_S\Big] \geq 0}_{\geq 0}$$

$$\geq X_S.$$