Martingales and stopping times

Last time: If X is a submartingale and $S \le T$ are two bounded stopping times, then

$$E[X_T \mid \mathcal{F}_S] \ge X_S \quad \text{a.s.}$$

Prop: Assume that X is a bounded submartingale and that the stopping times $S \le T$ are finite a.s. Then

$$E[X_T \mid \mathcal{F}_S] \ge X_S \quad \text{a.s.}$$

(and correspondingly for supermartingales ($\le$) and martingales ($=$))

Proof: It suffices to prove that if $\Delta \in \mathcal{F}_S$, then

$$\int_\Delta X_T \, dP \ge \int_\Delta X_S \, dP$$

For any $n \in \mathbb{N}$, the stopping times $T \wedge n$ and $S \wedge n$ are bounded, so by the previous result

$$(*) \qquad E[X_{T\wedge n} \mid \mathcal{F}_{S \wedge n}] \ge X_{S\wedge n}$$

If $m \le n$, then $\Delta \cap \{S = m\} \in \mathcal{F}_{S\wedge n}$. Hence

$$\int_{\Delta \cap \{S=m\}} \overset{\nearrow X_T}{X_{T\wedge n}} \, dP \ge \int_{\Delta \cap \{S=m\}} \overset{\nearrow X_S}{X_{S\wedge n}} \, dP$$

Take the limit as $n \to \infty$: Since X is bounded, the DCT gives

$$\int_{\Delta \cap \{S=m\}} X_T \, dP \ge \int_{\Delta \cap \{S=m\}} X_S \, dP$$

Adding these inequalities for $m = 1, \dots, N$, we get

$$\int_{\Delta \cap \{S\le N\}} X_T \, dP \ge \int_{\Delta \cap \{S\le N\}} X_S \, dP$$

i.e.

$$\int_\Delta \overbrace{\mathbb{1}_{\{S\le N\}} X_T}^{\uparrow X_T \text{ a.s.}} \, dP \ge \int_\Delta \overset{X_S \text{ a.s.}}{\overset{\uparrow}{\mathbb{1}_{\{S \le N\}} X}} \, dP$$

By DCT, we get

$$\int_\Delta X_T \, dP \ge \int_\Delta X_S \, dP \quad \text{for all } \Delta \in \mathcal{F}_S.$$

Hence $E[X_T \mid \mathcal{F}_S] \ge X_S$ a.s.

Example: Simple random walk. Martingale.

$T = \inf\{n : X_n(\omega) = 1\}$ → T is finite a.s.

$S = 0$

By the theorem, one would expect

$$E[X_T] \ge E[X_S] = 0.$$

Actually $E[X_T] = E[1] = \underline{\underline{1}}$

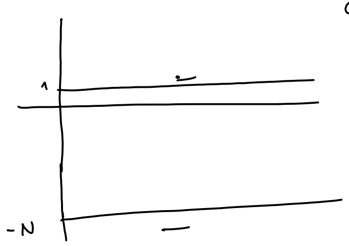
Problem: Neither T nor X are bounded.

Only expect to live for 20 000 years:

$T \wedge 20000$ $\qquad E[X_{T\wedge 20000}] = E[X_0] = 0.$

Limited amount: Can not loose more than N

$$T = \inf\{n : X_n(\omega) = 1 \text{ or } X_n(\omega) = -N\}$$

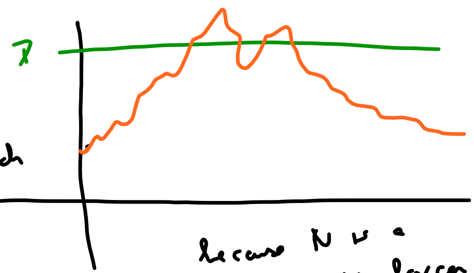


Change the martingale so that it is constant outside $[-N, 1]$.

$$E[X_T] = E[X_0] = 0$$

Martingale Maximal Inequality: Assume that M is a positive submartingale and $\lambda > 0$ and $N \in \mathbb{N}$.

$$\lambda P[\max_{1 \le n \le N} M_n \ge \lambda] \le E[M_N]$$



Proof: Let

$$T(\omega) = \begin{cases} \text{first } n^{\le N} \text{ such that } M_n(\omega) \ge \lambda & \text{if such an } n \text{ exists} \\ N & \text{otherwise.} \end{cases}$$

T is a stopping and hence

$$\lambda P[\max_{1 \le n \le N} M_n \ge \lambda] = \lambda P[M_T \ge \lambda] \overset{\text{Chebyshev}}{\le} E[M_T] \underset{\substack{\text{because } N \text{ is a} \\ \text{stopping larger than } T.}}{\le} E[M_N]$$

Problems

Exam 2019, pr 4

Page 274 : 8.2, 8.3, 8.5, 8.6, 8.7

Problem 4, 2019 : $\left.\begin{array}{l}X\\ Y\end{array}\right\}$ independent $\left.\begin{array}{l}F\\ G\end{array}\right\}$ distribution function

Then the distribution function of $X+Y$ is
$$K(x) = E[F(x-Y)] = \int F(x-y)\,dG(y)$$

a) Define $K(x) = E[F(x-Y)]$. Show that K is a distribution function: i.e.

K is increasing

K is right continuous

$\lim_{x\to\infty} K(x) = 1$, $\lim_{x\to-\infty} K(x) = 0$

K is increasing: $x_2 > x_1$

$$K(x_2) = E[\underset{\uparrow \text{inc}}{F(x_2-Y)}] \geq E[F(x_1-Y)] = K(x_1)$$

K is right continuous: $\lim_{y\downarrow x} K(y) = \lim_{y\downarrow x} E[\underbrace{F(y-Y)}_{F(x-Y)}] \overset{DCT}{=}$

$= E[\lim_{y\downarrow x} F(y-Y)] = E[F(x-Y)] = K(x)$.

$\lim_{x\to\infty} K(x) = \lim_{x\to\infty} E[\overbrace{F(x-Y)}^{1}] \overset{DCT}{=} E[\lim_{y\to\infty} F(x-Y)] = E[1] = 1$.

$\lim_{x\to-\infty} K(x) = 0$ is similar.

b) Assume that $Y = \sum_{i=1}^{\infty} a_i \mathbb{1}_{A_i}$ where the a_i's are distinct and $\{A_i\}$ is a measurable partition of Ω.

Show that if Y is independent of X, then the dist. fund of $X+Y$ is $K(x) = E[F(x-Y)]$

We have $K(x) = P[X+Y \leq x] = \sum_{i=1}^{\infty} P[Y = a_i \wedge X \leq x-a_i]$

$\overset{\text{indep}}{=} \sum_{i=1}^{\infty} P[Y=a_i]\underbrace{P[X \leq x-a_i]}_{F(x-a_i)} = \sum_{i=1}^{\infty} F(x-a_i)P[Y=a_i]$

$= E[F(x-Y)]$

$\hookrightarrow Y = \sum a_i \mathbb{1}_{A_i}$

c) Assume that Y is independent of X, and let
$Y_n = \sum \frac{k}{2^n} \mathbb{1}_{\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]}$. Hence $Y_n \uparrow Y$ pointwise

Show that
$$E[F(x-Y)] = \lim_{n\to\infty} E[\underbrace{F(x-Y_n)}_{\text{decreases to } F(x-Y)}]$$

Hence

$\lim_{n\to\infty} E[F(x-Y_n)] \overset{DCT}{\longrightarrow} E[\lim_{n\to\infty} F(x-Y_n)] = E[F(x-Y)]$

d) Show that the distribution function H of $X+Y$ is given by $H(x) = E[F(x-Y)]$ for all Y independent of X.

Y_n are independent of X, and hence by b) the distribution function of $X+Y_n$ is $H_n(x) = E[F(x-Y_n)] \overset{c}{\to} E[F(x-Y)]$

Since $X+Y_n \to X+Y$ pointwise, then $X+Y_n \Rightarrow X+Y$,

Hence $H_n(x) \to H(x)$ at all continuity point.

$E[F(x-Y_n)] \to E[F(x-Y)]$ everywhere

Thus $\underset{\substack{\uparrow\\ X+Y}}{H(x)} = E[F(x-Y)]$

$H(x) = \int F(x-y)\,dG(y)$,

$\int f(y)\,dG(y) = E[f(Y)]$

8.2 If $\mathcal{G}_1, \mathcal{G}_2$ are independent

X independent of $\mathcal{G}$

$E[X|\mathcal{G}] = E[X]$

$$E[\underbrace{E[X|\mathcal{G}_1]}_{\text{independent}} | \mathcal{G}_2] = E[X]$$

$$E[\underbrace{E[X|\mathcal{G}_1]} | \mathcal{G}_2] = E[E[X|\mathcal{G}_1]] = E[X].$$

8.3 Show that if $\mathcal{G}$ is trivial, then

$$E[X|\mathcal{G}] = E[X].$$

Must show that

(i) $E[X]$ is $\mathcal{G}$-measurable ✓

(ii) $E[X]$ is integrable ✓

(iii) $\int_\Lambda E[X]\,dP = \int_\Lambda X\,dP$ ✓

a) $P(\Lambda) = 0$: $0 = 0$ ✓

b) $P(\Lambda) = 1$: $\int_\Lambda E[X]\,dP = E[X]P(\Lambda) = E[X]$

$$E[X] = \int_\Omega X\,dP = \int_\Lambda X\,dP + \int_{\Lambda^c} X\,dP = \int_\Lambda X\,dP$$

$P(\Lambda^c)$

8.5 X, Y, XY are integrable. Then

$$E[XY] = E[X\,E[Y|\sigma(X)]]$$

Proof: $E[XY] = E[E[XY|\sigma(X)]] = E[X\,E[Y|\sigma(X)]]$

$\sigma(X)$-measurable

8.6 X, Y integrable, $E[X] = E[Y]$, Y is $\mathcal{G}$-measurable

$X - Y$ is independent of $\mathcal{G}$.

Prove that $Y = E[X|\mathcal{G}]$.

Proof: $E[X|\mathcal{G}] = E[(X-Y)+Y|\mathcal{G}]$

$$= E[\underbrace{X-Y}_{\text{independent of } \mathcal{G}}|\mathcal{G}] + E[\underbrace{Y}_{\mathcal{G}\text{-meas}}|\mathcal{G}] = E[X-Y] + Y = E[X] - E[Y] + Y$$

$= Y$.