Fourier inversion

Characteristic function:

$$\varphi(t) = E[e^{itX}] \overset{\text{if density}}{=} \int_{-\infty}^{\infty} e^{itx} f(x)\,dx = \hat{f}(t)$$

Fourier inversion formula:

$$f(x) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \varphi(t)\, e^{-itx}\,dt$$

If $\varphi(t)$ is integrable, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t)\, e^{-itx}\,dt$$

Example: $X = a$ (constant)

$$\varphi(t) = E[e^{itX}] = e^{ita} = \cos ta + i \sin ta.$$

Motivation for inversion formula: Consider

$$I_T = \frac{1}{2\pi}\int_{-T}^{T} \varphi(t)\, e^{-itx}\,dt \qquad \varphi(t) = \int_{-\infty}^{\infty} e^{iyt} f(y)\,dy$$

$$= \frac{1}{2\pi}\int_{-T}^{T}\left(\int_{-\infty}^{\infty} e^{ity} f(y)\,dy\right) e^{-itx}\,dt$$

$$= \frac{1}{2\pi}\int_{-T}^{T}\left(\int_{-\infty}^{\infty} e^{it(y-x)} f(y)\,dy\right) dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-T}^{T} e^{it(y-x)} f(y)\,dt\right] dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\left(\int_{-T}^{T} e^{it(y-x)}\,dt\right) dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\left[\frac{e^{it(y-x)}}{i(y-x)}\right]_{t=-T}^{T} dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\left[\frac{e^{iT(y-x)} - e^{-iT(y-x)}}{i(y-x)}\right] dy$$

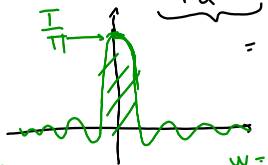
$$= \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} f(y)\,\frac{\cancel{2}}{y-x}\overbrace{\left[\frac{e^{iT(y-x)} - e^{-iT(y-x)}}{2i}\right]}^{\sin[T(y-x)]} dy \qquad \boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}$$

$$= \int_{-\infty}^{\infty} f(y)\,\underbrace{\frac{\sin[T(y-x)]}{\pi(y-x)}}_{g_T(y-x)}\,dy \qquad g_T(u) = \frac{\sin Tu}{\pi u}$$

$$= \int_{-\infty}^{\infty} f(y)\,\underbrace{g_T(y-x)}_{\text{contribution when } y\approx x}\,dy \to f(x)$$

Take a look at $g_T(u)$

$$g_T(0) = \underbrace{\frac{\sin Tu}{Tu}}\cdot\frac{T}{\pi} = \frac{T}{\pi}$$



Also

$$\int_{-\infty}^{\infty} g_T(u)\,du = \int_{-\infty}^{\infty} \frac{\sin Tu}{\pi u}\,du \qquad \begin{aligned} w &= Tu \\ dw &= T\,du \end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{\sin w}{\pi \frac{w}{\cancel{T}}}\,\frac{dw}{\cancel{T}} = \int_{-\infty}^{\infty} \frac{\sin w}{\pi w}\,dw$$

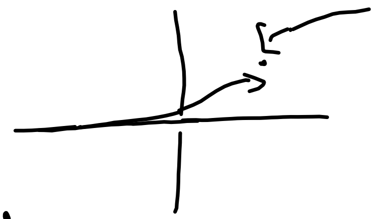
$$= \underline{1}$$

<u>Theorem:</u> Assume that X is a random variable with a density f and a characteristic function φ. Then

$$f(x) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \varphi(t) e^{-itx}\, dt$$

If φ is integrable, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t) e^{-itx}\, dt.$$



If F is a distribution function, define

$$\bar{F}(x) = \frac{F(x) + F(x-)}{2}$$

<u>Levy's Inversion Formula:</u> Let X be a random variable with characteristic function φ and distribution function F. For $a < b$,

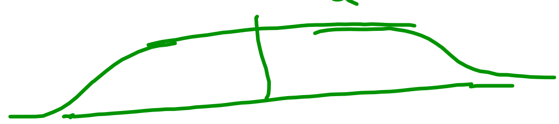
$$\bar{F}(b) - \bar{F}(a) = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \underbrace{\frac{e^{-ibt} - e^{-iat}}{-2\pi i t}\varphi(t)}_{h(t)} e^{-\frac{\varepsilon^2 t^2}{2}}\, dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} \underbrace{\frac{e^{-ibt} - e^{-iat}}{-2\pi i t}\varphi(t)}_{h(t)}\, dt.$$

$$\int_{-T}^{T} \underbrace{\frac{e^{-ibt} - e^{-iat}}{-2\pi i t}}_{h(t)} \varphi(t)\, dt = \int_{-\infty}^{\infty} \underbrace{\mathbb{1}_{[-T,T]}} h(t)\, dt$$

$$\int_{-\infty}^{\infty} h(t) e^{-\frac{\varepsilon^2 t^2}{2}}\, dt$$

$e^{-\frac{\varepsilon^2 t^2}{2}}$



Let ξ be a $N(0,1)$ random variable independent of X and define

$$X_\varepsilon = X + \varepsilon\xi \quad \text{for } \varepsilon > 0.$$

Then X_ε is "smooth" random variable with a density f_ε. Then

$$P[a < X_\varepsilon < b] = \int_a^b f_\varepsilon(x)\,dx = \int_a^b \underbrace{\Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_\varepsilon(t)e^{-itx}\,dt\Big]}_{f_\varepsilon(x)}dx$$

Note $\varphi_\varepsilon(t) = \varphi(t)e^{-\frac{\varepsilon^2 t^2}{2}}$ (integrable)

$$\frac{1}{2\pi}\int_a^b\Big[\int_{-\infty}^{\infty}\varphi(t)e^{-\frac{\varepsilon^2t^2}{2}}e^{-itx}\,dt\Big]\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\int_a^b \varphi(t)e^{-\frac{\varepsilon^2t^2}{2}}e^{-itx}\,dx\Big]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)e^{-\frac{\varepsilon^2t^2}{2}}\Big[\int_a^b e^{-itx}\,dx\Big]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)e^{-\frac{\varepsilon^2t^2}{2}}\Big[\frac{e^{-itx}}{-it}\Big]_{x=a}^{x=b}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)e^{-\frac{\varepsilon^2t^2}{2}}\Big[\frac{e^{-itb}-e^{-ita}}{-it}\Big]dt$$

$$= \int_{-\infty}^{\infty}\frac{e^{-itb}-e^{-ita}}{-2\pi it}\varphi(t)e^{-\frac{\varepsilon^2t^2}{2}}\,dt$$

Hence

$$\lim_{\varepsilon\to 0} P[a<X_\varepsilon<b] = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{e^{-itb}-e^{-ita}}{-2\pi it}\varphi(t)e^{-\frac{\varepsilon^2t^2}{2}}\,dt$$

What happens with the limit on the left?

$$\lim_{\varepsilon\to 0}P[a<X_\varepsilon<b] = \lim_{\varepsilon\to 0}E\big[\mathbb{1}_{(a,b)}(X_\varepsilon)\big]$$

$$\overset{DCT}{=} E\Big[\underbrace{\lim_{\varepsilon\to 0}\mathbb{1}_{(a,b)}(X_\varepsilon)}\Big] = \bar{F}(b) - \bar{F}(a)$$

1 if $X \in (a,b)$

if $X = a$, then we are in the set with prob $\frac{1}{2}$

if $X = b$, ——— ''

Hence

$$\bar{F}(b) - \bar{F}(a) = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{e^{-ibt}-e^{-iat}}{-2\pi it}\varphi(t)e^{-\frac{\varepsilon^2t^2}{2}}\,dt.$$

Corollary: Two different distributions cannot have the same characteristic function.

Proof: Assume not, and F, G be two distributions with the same char. funct. φ

Then $\bar{F}(b) - \bar{F}(a) = \bar{G}(b) - \bar{G}(a)$

Letting $a \to -\infty$, we get $\bar{F}(b) = \bar{G}(b)$. But at all point x, $F(x) = \lim_{y\downarrow x}\bar{F}(y) = \lim_{y\to x} F(y)$. The same for G.

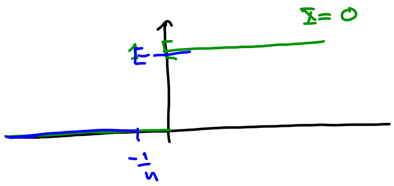
Convergence in distribution

Recall: X_n converges to X in distribution if

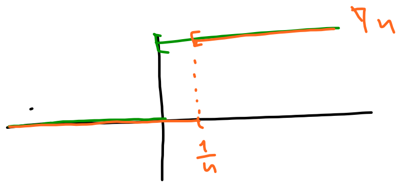
$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ in all points x where F_X is continuous.

Example: $X_n = -\frac{1}{n}$ should converge to $X = 0$

$Y_n = \frac{1}{n}$ —" — $X = 0$



$F_{X_n}(x) \to F_X(x)$ for all x



$F_{Y_n}(x) \to F_X(x)$ for all $x \neq 0$.

$F_X(x) = P[X < x]$

Lemma: Assume that $\{X_n\}$ is a sequence of random variable with distribution function F_n. If F is the distribution function of X and

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for all x in a dense set of points D, then X_n converges to X in distribution.

Proof: Let x be a continuity point of F. We need to show that $F_n(x) \to F(x)$. Let $\varepsilon > 0$ be given. Since ~~D~~ D is dense and F is cont at x, we can find points $a, b \in D$ such that

$$a < x < b$$

such that $F(a) > F(x) - \varepsilon$ and $F(b) < F(x) + \varepsilon$

But then

$$\limsup_{n\to\infty} F_n(x) \leq \limsup F_n(b) = F(b) < F(x) + \varepsilon$$

$$\liminf_{n\to\infty} F_n(x) \geq \liminf F_n(a) = F(a) > F(x) - \varepsilon$$

Hence $\limsup F_n(x) = \liminf F_n(x) = F(x)$ and thus $\lim_{n\to\infty} F_n(x) = F(x)$.

$E[f(X_n)] \to E[f(X)]$ for all bounded cont f.