Convergence in distribution

Def: $X_n \to X$ in distribution [if] $F_n(x) \to F(x)$ at all points x where F is continuous.

Assume that [F] is continuous at a, b. Then

$$P[a < X_n \le b] = F_n(b) - F_n(a) \to F(b) - F(a) = P[a < X \le b]$$

hence

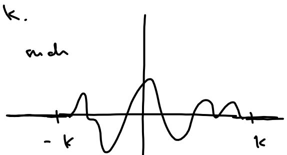
$$\lim_{n\to\infty} P[a < X_n \le b] = P[a < X \le b]$$

Theorem: $\{X_n\}$ converges to X in distribution if and only if

$$(*) \qquad \lim_{n\to\infty} E[f(X_n)] = E[f(X)]$$

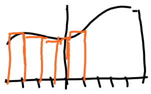
for all bounded, continuous function $f : \mathbb{R} \to \mathbb{R}$.

Proof: Assume that $X_n \to X$ in distribution. We are first going to prove $(*)$ for continuous function with compact support, i.e. there is a number $k \in \mathbb{R}$ such that $f(x) = 0$ when $|x| \ge k$.



Given $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $|u - v| < \delta$, then $|f(u) - f(v)| < \varepsilon$ (since cont. fund. on compact intervals are uniformly continuous). Hence we can partition $[-k, k]$ into subintervals $(a_0, a_1], (a_1, a_2], \dots, (a_{n-1}, a_n]$ with lengths less than δ such that $a_0, a_1, \dots, a_n$ are continuity points for F (the dist. function of X).

Let $\psi(x) = \sum_{i=1}^{n} f(a_i)\, 1\!\!1_{(a_{i-1}, a_i]}$



Note that $|\psi(x) - f(x)| < \varepsilon$. Note that

$$|E[f(X_n)] - E[f(X)]| = |E[f(X_n) - f(X)]|$$

$$= |E[f(X_n) - \psi(X_n) + \psi(X_n) - \psi(X) + \psi(X) - f(X)]|$$

$$\le E[\underbrace{|f(X_n) - \psi(X_n)|}_{<\varepsilon}] + |E[\psi(X_n) - \psi(X)]| + E[\underbrace{|\psi(X) - f(X)|}_{<\varepsilon}]$$

$$\le 2\varepsilon + \left| E\left[\sum f(a_i)\, 1\!\!1_{(a_{i-1}, a_i]}(X_n) - \sum f(a_i)\, 1\!\!1_{(a_{i-1}, a_i]}(X)\right]\right|$$

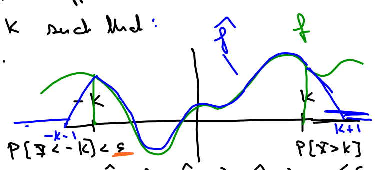
$$\le 2\varepsilon + \sum_{i=1}^{n} f(a_i)\underbrace{\left[P[a_{i-1} < X_n \le a_i] - P[a_{i-1} < X \le a_i]\right]}_{\downarrow 0}$$

Letting n go to infinity, we see that $|E[f(X_n)] - E[f(X)]|$ gets less that ε, and as $\varepsilon > 0$ is arbitrary, we must have

$$E[f(X_n)] \to E[f(X)].$$

Let us next look at a function f which is bounded, continuous, but not compact support.

Given $\varepsilon > 0$, choose K such that: and define $\hat{f}$ as shown.



$$|E[f(X_n) - f(X)]| = E[f(X_n) - \hat{f}(X_n) + \hat{f}(X_n) - \hat{f}(X) + \hat{f}(X) - f(X)]$$

$$\le \underbrace{2M\, P[X_n < -K] + 2M\, P[X_n > K]}_{4M\varepsilon}$$

$$+ E[\hat{f}(X_n) - \hat{f}(X)] + \underbrace{2M\, P[X < -K] + 2M\, P[X > K]}_{4M\varepsilon}$$

$$\le 8M\varepsilon + E[\hat{f}(X_n) - \hat{f}(X)] \to 0.$$

Hence for large n, $E[f(X_n) - f(X)] \le 8M\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this means $\lim E[f(X_n)] = E[f(X)]$.

Assume now that

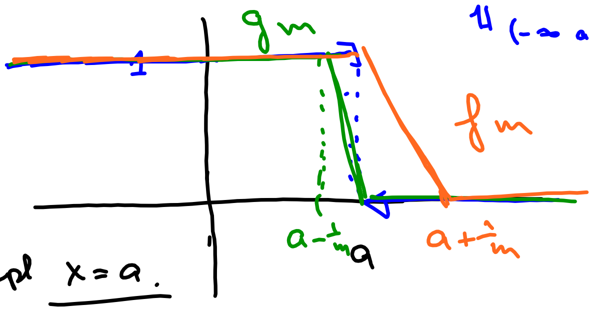
$(*)$ $\lim_{n\to\infty} E[f(X_n)] = E[f(X)]$ for all bounded, cont. f.

Assume that a is a continuity point for X. Then to prove that $\lim_{n\to\infty} F_n(a) = F(a)$

Note that $F_n(a) = P[X_n \le a] = E[\mathbb{1}_{(-\infty,a]}(X_n)]$

$F(a) = P[X \le a] = E[\mathbb{1}_{(-\infty,a]}(X)]$

Approx:



$g_m(x) \le \mathbb{1}_{(-\infty,a]}(x) \le f_m(x)$

$g_m(x) \to \mathbb{1}_{(-\infty,a]}(x)$ for all x except $\underline{x=a}$.

$f_m(x) \to \mathbb{1}_{(-\infty,a]}(x)$ for all x.

$$\liminf_n \underbrace{E[g_m(X_n)]}_{} \le \liminf_{n\to\infty} E[\mathbb{1}_{(-\infty,a]}(X_n)] \le \limsup_n E[\mathbb{1}_{(-\infty,a]}(X_n)]$$

$$\le \limsup_n E[f_m(X_n)]$$

$(*)$ $\|$ $E[g_m(X)]$ ... $\to \mathbb{1}_{(-\infty,a]}(X)$ a.s.

$\downarrow m\to\infty$

$E[\mathbb{1}_{(-\infty,a]}(X)]$

$\|$ $(*)$ $E[f_m(X)]$ ... $\downarrow m\to\infty$

$E[\mathbb{1}_{(-\infty,a]}(X)]$

DCT

Hence

$$E[\mathbb{1}_{(-\infty,a]}(X)] \le \liminf E[\mathbb{1}_{(-\infty,a]}(X_n)]$$
$$\le \limsup E[\mathbb{1}_{(-\infty,a]}(X_n)] \le E[\mathbb{1}_{(-\infty,a]}(X)]$$

Hence $\lim E[\mathbb{1}_{(-\infty,a]}(X_n)] = E[\mathbb{1}_{(-\infty,a]}(X)]$

which is what we had to prove!

Trial Exam 1, 2019, Prob 4

a) $\{x_n\}$ sequence of numbers

$$S_n = \frac{x_1 + x_2 + \cdots + x_n}{\sqrt{n}}$$

$$S_n^k = \frac{x_k + x_{k+1} + \cdots + x_n}{\sqrt{n}}$$

Show that: $\limsup_{n\to\infty} S_n = \limsup_{n\to\infty} S_n^k$

$$|S_n - S_n^k| = \left|\frac{x_1 + x_2 + \cdots + x_{k-1}}{\sqrt{n}}\right| \leq \frac{(k-1)M}{\sqrt{n}} \quad \text{where } M = \max\{|x_1|, |x_2|, \ldots, |x_{k-1}|\}$$

ε

Given $\varepsilon > 0$, I can always find $m \in \mathbb{N}$ such that

$|S_n - S_n^k| \leq \varepsilon$ when $n \geq m$.

$$\left|\sup_{n\geq m} S_n - \sup_{n\geq m} S_n^k\right| \leq \varepsilon.$$

Thus

$$\left|\lim_{m\to\infty}\sup_{n\geq m} S_n - \lim_{m\to\infty}\sup_{n\geq m} S_n^k\right| \leq \varepsilon$$

$$|\limsup S_n - \limsup S_n^k| < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we must have

$$\limsup S_n = \limsup S_n^k$$

b) Assume now that $\{X_n\}$ is a sequence of independent random variables and put

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}$$

Show that for any Borel set B, the set

$A = \{\omega : \limsup S_n \in B\}$

is a tail event, i.e. $A \in \mathcal{F}_\infty^* = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n^*$

where $\mathcal{F}_n^* = \sigma(X_{n+1}, X_{n+2}, \ldots)$

$$A = \{\omega : \limsup S_n \in B\} = \{\omega : \limsup S_n^k \in B\} \in \mathcal{F}_{k-1}^*$$

where $S_n^k = \dfrac{X_k + X_{k+1} + \cdots + X_n}{\sqrt{n}}$

Since this works for any k, we must have $A \in \mathcal{F}_\infty^*$

§10 If $X_n \to 0$ a.e., then $\frac{X_1+X_2+\dots+X_n}{n} \to 0$ a.e.

Show that there are independent sequences $\{X_n\}$ s.t. $X_n \to 0$ in prob., but $\frac{X_1+\dots+X_n}{n}$ does not go to 0 in prob.

First part: It is known that if $x_n \to a$, then

$$\underbrace{\frac{x_1+x_2+\dots+x_n}{n} \to a}_{\text{Cesaro convergence.}}$$

Second part: Assume that $\{X_n\}$ are independent r.v. with

$$X_n = \begin{cases} 2^n & \text{with prob } \frac{1}{n} \\ 0 & \text{with prob } 1-\frac{1}{n}, \end{cases}$$

$\sum P(B_n) = \sum \frac{1}{n} = \infty$. BC. with prob 1, there will be infinitely many success.

Claim $X_n \to 0$ in prob. (from the definition)

$$\frac{X_1+\dots+X_n}{n} \not\to 0 \text{ in prob.}$$

If ω is in the set of prob 1 when there are infinitely many success,

$$\frac{X_1+\dots+X_{n-1}+2^n}{n} \geq \frac{2^n}{n} > 1$$

we will have infinitely situation of this kind

No convergence in prob.

5.11 $X_1, \dots, X_n$ independent, $E[X_i]$, $E[X_i^2] < \infty$.

$S_k = X_1 + \dots + X_k$

$\Lambda = \{\omega : \max_{k \le n} |S_k| \ge \varepsilon\}$

$\Lambda = \Lambda_1 \cup \Lambda_2 \cup \dots \cup \Lambda_n$ disjoint

$\Lambda_k = \{\omega : |S_k| \ge \varepsilon$ and $|S_i| < \varepsilon$ for all $i < k\}$

Show that $\int_{\Lambda_k} S_n^2 \, dP \ge \int_{\Lambda_k} S_k^2 \, dP$

$$\int_{\Lambda_k} S_n^2 \, dP = \int_{\Lambda_k} (S_k + (S_n - S_k))^2 \, dP$$

$$= \int_{\Lambda_k} S_k^2 \, dP + \int \underbrace{\mathbb{1}_{\Lambda_k} 2S_k}_{\sigma(X_1, \dots, X_k)} \underbrace{(S_n - S_k)}_{\sigma(X_{k+1}, \dots, X_n)} \, dP + \int_{\Lambda_k} (S_n - S_k)^2 \, dP$$

$$\underbrace{}_{= 0}$$

$$= \int_{\Lambda_k} S_k^2 \, dP + \int_{\Lambda_k} (S_n - S_k)^2 \, dP \ge \int_{\Lambda_k} S_k^2 \, dP$$

$$P[\max_{k \le n} |S_k| \ge \varepsilon] \le \frac{1}{\varepsilon^2} E[S_n^2] \quad \text{Kolmogorov ineq.}$$

$$E[S_n^2] \ge \int_{\Lambda} S_n^2 \, dP = \sum_{k=1}^{n} \int_{\Lambda_k} S_n^2 \, dP \ge \sum_{k=1}^{n} \int_{\Lambda_k} S_k^2 \, dP$$

$$\ge \sum_{k=1}^{n} \varepsilon^2 P(\Lambda_k) = \varepsilon^2 P(\Lambda)$$

$$= \varepsilon^2 P[\max_{k \le n} S_k \ge \varepsilon]$$